Tractable Local Equilibria in Non-Concave Games

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Abstract

While Online Gradient Descent and other no-regret learning procedures are known to efficiently converge to coarse correlated equilibrium in games where each agent's utility is concave in their own strategy, this is not the case when the utilities are non-concave, a situation that is common in machine learning applications where the agents' strategies are parameterized by deep neural networks, or the agents' utilities are computed by a neural network, or both. Indeed, non-concave games present a host of game-theoretic and optimization challenges: (i) Nash equilibria may fail to exist; (ii) local Nash equilibria exist but are intractable; and (iii) mixed Nash, correlated, and coarse correlated equilibria have infinite support in general, and are intractable. To sidestep these challenges we propose a new solution concept, termed $(\varepsilon, \Phi(\delta))$-local equilibrium, which generalizes local Nash equilibrium in non-concave games, as well as (coarse) correlated equilibrium in concave games. Importantly, we show that two instantiations of this solution concept capture the convergence guarantees of Online Gradient Descent and no-regret learning, which we show efficiently converge to this type of equilibrium in non-concave games with smooth utilities.

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1 Introduction

Von Neumann's celebrated minimax theorem establishes the existence of Nash equilibrium in all two-player zero-sum games where the players' utilities are continuous as well as *concave* in their own strategy [v. Neumann, 1928].[1] This assumption that players' utilities are concave, or quasi-concave, in their own strategies has been cornerstone for the development of equilibrium theory in Economics, Game Theory, and a host of other theoretical and applied fields that make use of equilibrium concepts. In particular, (quasi-)concavity is key for showing the existence of many types of equilibrium, from generalizations of min-max equilibrium [Fan, 1953, Sion, 1958] to competitive equilibrium in exchange economies [Arrow and Debreu, 1954, McKenzie, 1954], mixed Nash equilibrium in finite normal-form games [Nash Jr, 1950], and, more generally, Nash equilibrium in (quasi-)concave games [Debreu, 1952, Rosen, 1965].

Not only are equilibria guaranteed to exist in concave games, but it is also well-established—thanks to a long line of work at the interface of game theory, learning and optimization whose origins can be traced to Dantzig's work on linear programming [George B. Dantzig, 1963], Brown and Robinson's work on fictitious play [Brown, 1951, Robinson, 1951], Blackwell's approachability theorem [Blackwell, 1956] and Hannan's consistency theory [Hannan, 1957]—that several solution concepts are efficiently computable both centrally and via decentralized learning dynamics. For instance, it is well-known that the learning dynamics produced when the players of a game iteratively update their strategies using no-regret learning algorithms, such as online gradient descent, is guaranteed to converge to Nash equilibrium in two-player zero-sum concave games, and to coarse correlated equilibrium in multi-player general-sum concave games [Cesa-Bianchi and Lugosi, 2006]. The existence of such simple decentralized dynamics further justifies using these solution concepts to predict the outcome of real-life multi-agent interactions where agents deploy strategies, obtain feedback, and use that feedback to update their strategies.

While (quasi-)concave utilities have been instrumental in the development of equilibrium theory, as described above, they are also too restrictive an assumption. Several modern applications and outstanding challenges in Machine Learning, from training Generative Adversarial Networks (GANs) to Multi-Agent Reinforcement Learning (MARL) as well as generic multi-agent Deep Learning settings where the agents' strategies are parameterized by deep neural networks or their utilities are computed by deep neural networks, or both, give rise to games where the agents' utilities are *non-concave* in their own strategies. We call these games *non-concave*, following Daskalakis [2022].

Unfortunately, classical equilibrium theory quickly hits a wall in non-concave games. First, Nash equilibria are no longer guaranteed to exist. Second, while mixed Nash, correlated and coarse correlated equilibria do exist—under convexity and compactness of the strategy sets [Glicksberg, 1952], which we have been assuming all along in our discussion so far, they have infinite support, in general [Karlin, 2014]. Finally, they are computationally intractable; so, a fortiori, they are also intractable to attain via decentralized learning dynamics.

[1] Throughout this paper, we model games using the standard convention in Game Theory that each player has a utility function that they want to maximize. This is, of course, equivalent to modeling the players as loss minimizers, a convention more common in learning. When we say that a player's utility is concave (respectively non-concave) in their strategy, this is the same as saying that the player's loss is convex (respectively non-convex) in their strategy.

To address the limitations associated with classical, global equilibrium concepts, a natural approach is to focus on developing equilibrium concepts that guarantee local stability instead. One definition of interest is the strict local Nash equilibrium, wherein each player's strategy corresponds to a local maximum of their utility function, given the strategies of the other players. Unfortunately, a strict local Nash equilibrium may not always exist, as demonstrated in Example 3. Furthermore, a weaker notion—the second-order local Nash equilibrium, where each player has no incentive to deviate based on the second-order Taylor expansion estimate of their utility, is also not guaranteed to exist as illustrated in Example 3. What's more, it is NP-hard to check whether a given strategy profile is a strict local Nash equilibrium or a second-order local Nash equilibrium, as implied by the result of Murty and Kabadi [1987].[2] Finally, one can consider *local Nash equilibrium* (see Definition 1), a first order stationary solution, which is guaranteed to exist [Daskalakis et al., 2021b]. Unlike non-convex optimization, where targeting first-order local optima sidesteps the intractability of global optima, this first-order local Nash equilibrium has been recently shown to be intractable, even in two-player zero-sum non-concave games with joint feasibility constraints [Daskalakis et al., 2021b].[3] See Table 1 for a summary of solution concepts in non-concave games.

More broadly, the study of local equilibrium concepts that are guaranteed to exist in non-concave games has received a lot of attention in recent years—see e.g. [Ratliff et al., 2016, Hazan et al., 2017, Daskalakis and Panageas, 2018, Jin et al., 2020, Daskalakis et al., 2021b]. However, in terms of computing the local equilibrium concepts that have been proposed, existing results are restricted to sequential two-player zero-sum games [Mangoubi and Vishnoi, 2021]; or only establish local convergence guarantees for learning dynamics—see e.g. [Daskalakis and Panageas, 2018, Wang et al., 2020, Fiez et al., 2020]; or only establish asymptotic convergence guarantees—see e.g. [Daskalakis et al., 2023]; or involve solution concepts that are non-standard in that their local stability is not with respect to a distribution over strategy profiles [Hazan et al., 2017]. In view of the importance of non-concave games in emerging ML applications and the afore-described state-of-affairs, our investigation in this paper is motivated by the following broad and largely open question:

Question from [Daskalakis, 2022]: *Is there a theory of non-concave games? What solution concepts are meaningful, universal, and tractable?*

1.1 Contributions

In this paper, we answer the question raised by Daskalakis [2022] by proposing a new, general, local equilibrium concept, as well as two concrete instantiations of this concept, both of which are game-theoretically meaningful, universal, and computationally tractable. Importantly, we show that simple decentralized learning dynamics, e.g., the dynamics induced when each player updates their strategy using online gradient descent (GD), efficiently converges to our equilibrium concepts. Throughout the paper, we focus on differentiable games whose strategy sets are subsets of $\mathbb{R}^d$ and

[2] Murty and Kabadi [1987] shows that checking whether a point is a local maximum of a multi-variate quadratic function is NP-hard.

[3] In general sum games, it is not hard to see that the intractability results [Daskalakis et al., 2009, Chen et al., 2009] for computing *global* Nash equilibria in bimatrix games imply intractability for computing *local* Nash equilibria.

Table 1: A comparison between different solution concepts in multi-player non-concave games. We include definitions of Nash equilibrium, mixed Nash equilibrium, (coarse) correlated equilibrium, strict local Nash equilibrium, and second-order local Nash equilibrium in Appendix A. We also give a detailed discussion on existence and complexity of these solution concepts in Appendix A. (*): the computation complexity of computing an approximate equilibrium or checking its existence and the representation complexity of an exact equilibrium.

Solution Concept	Incentive Guarantee	Existence	Complexity*
Nash equilibrium	Global optimality	✗	NP-hard
Mixed Nash equilibrium		✓	NP-hard, infinite support
(Coarse) Correlated equilibrium		✓	NP-hard, infinite support
Strict local Nash equilibrium	Strict local optimality	✗	NP-hard
Second-order local Nash equilibrium	Second-order stability	✗	NP-hard
Local Nash equilibrium	First-order stability	✓	PPAD-hard
Local Φ-equilibrium		✓	Reduces to Φ-regret minimization against a sequence of linear losses
Local Φ_{Int}-equilibrium		✓	Tractable via no-external regret alg
Local Φ_{Proj}-equilibrium		✓	Tractable via GD / OG

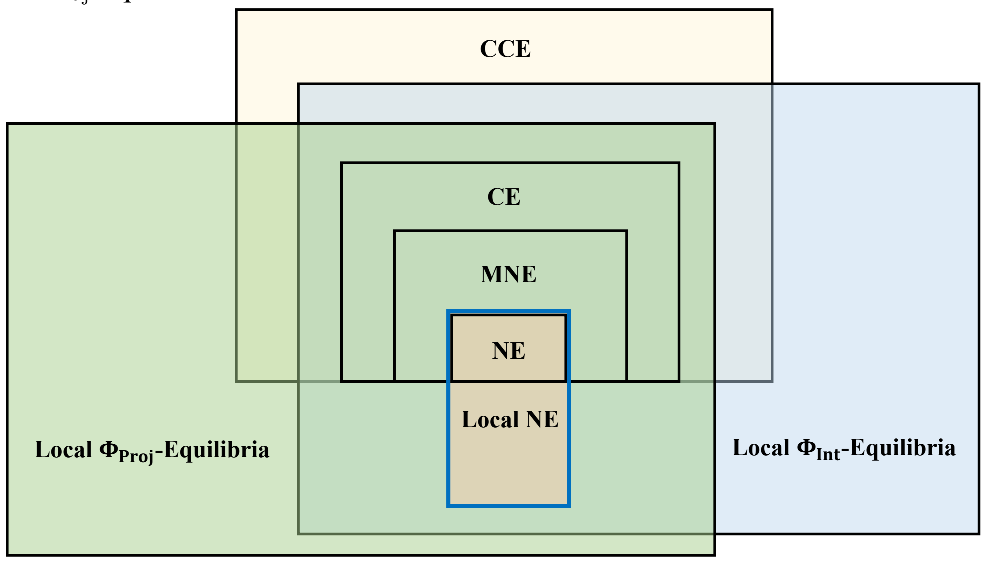
have G-Lipschitz and L-smooth (but not necessarily concave) utility functions (Assumption 1). Our contributions are as follows.

$(\varepsilon, \Phi(\delta))$-Local Equilibrium. A common way to sidestep the computational intractability of Nash equilibrium [Daskalakis et al., 2009, Chen et al., 2009] is to introduce *correlation* among the agents' strategies. Our local equilibrium concept uses this approach.[4] It is a joint distribution over $\Pi_{i=1}^n \mathcal{X}_i$, the Cartesian product of all players' strategy sets, and is defined in terms of a set, $\Phi^{\mathcal{X}_i}(\delta)$, of *$\delta$-local strategy modifications*, for each player i. The set $\Phi^{\mathcal{X}_i}(\delta)$ contains functions mapping $\mathcal{X}_i$ to itself, and it satisfies that, for all $\phi_i \in \Phi^{\mathcal{X}_i}(\delta)$ and all $x \in \mathcal{X}_i$: $\|\phi_i(x) - x\| \leq \delta$. In terms of $\Phi(\delta) = \Pi_{i=1}^n \Phi^{\mathcal{X}_i}(\delta)$, we propose the notion of $(\varepsilon, \Phi(\delta))$-*local equilibrium* to be a distribution over joint strategy profiles such that no player i can increase their expected utility by more than ε by applying any strategy modification function $\phi_i \in \Phi^{\mathcal{X}_i}(\delta)$ to the strategy sampled for them by the joint distribution. Clearly, the larger the set $\Phi(\delta)$, the stronger the incentive guarantee that the corresponding $(\varepsilon, \Phi(\delta))$-*local equilibrium* provides. $(\varepsilon, \Phi(\delta))$-local equilibrium generalizes the notion of local Nash equilibrium, since any (ε, δ)-local Nash equilibrium is, in fact, an $(\varepsilon, \Phi(\delta))$-local equilibrium, for any choice of $\Phi(\delta)$. This also guarantees the existence of $(\varepsilon, \Phi(\delta))$-local equilibrium in the regime where $\delta \leq \sqrt{2\varepsilon/L}$, which we refer to as the *local regime*, as (ε, δ)-local Nash equilibria are guaranteed to exist in the same regime [Daskalakis et al., 2021b].

We instantiate our $(\varepsilon, \Phi(\delta))$-local equilibrium concept by considering two natural choices for strategy modifications: (i) In the first instantiation, each player's set of local strategy modifications, denoted by $\Phi^{\mathcal{X}}_{\mathrm{Int},\Psi}(\delta)$, where $\mathcal{X}$ is $\mathcal{X}_i$ for player i, contains all deviations that *interpolate* between the input strategy and a modification suggested by $\psi \in \Psi$, which is a set of strategy modifications

[4] Our local equilibrium concept can be viewed as generalization of (coarse) correlated equilibrium from concave games to non-concave games. It is mostly useful in multi-agent systems where correlation across agents' strategies is natural, e.g. market dynamics, cooperation between self-driving cars, and interactions in social networks.

Figure 1: The relationship between different solution concepts in non-concave games. Specifically, we have (1) NE $\subseteq$ Local NE; (2) NE $\subseteq$ MNE $\subseteq$ CE $\subseteq$ CCE; (3) CE $\subseteq$ Local Φ_{Int}-equilibria and CE $\subseteq$ Local Φ_{Proj}-equilibria.



from $\mathcal{X}$ to $\mathcal{X}$. Formally, each element $\phi_{\lambda,\Psi}(x)$ of $\Phi^{\mathcal{X}}_{\mathrm{Int},\Psi}(\delta)$ can be represented as $(1-\lambda)x + \lambda\psi(x)$ for some $\psi \in \Psi$ and $\lambda \le \delta/D_{\mathcal{X}}$ ($D_{\mathcal{X}}$ is the diameter of $\mathcal{X}$). (ii) In the second instantiation, each player's set of local strategy modifications, denoted by $\Phi^{\mathcal{X}}_{\mathrm{Proj}}(\delta)$, contains all deviations that attempt a small step from their input in a fixed direction and project if necessary, namely are of the form $\phi_v(x) = \Pi_{\mathcal{X}}[x - v]$, where $\|v\| \le \delta$ and $\Pi_{\mathcal{X}}$ stands for the L_2-projection onto $\mathcal{X}$. [5] We draw a Venn diagram to illustrate the relationship of the two proposed local solution notions and other common solution concepts in Figure 1.

Local Φ-Regret Minimization. To efficiently compute an $(\varepsilon, \Phi(\delta))$-local equilibrium in non-concave games, we draw a connection between online learning and game theory. We show that online learning algorithms that achieve low $\Phi^{\mathcal{X}}(\delta)$-regret against adversarial sequences of *convex* loss functions can be employed to converge to a $(\varepsilon, \Phi(\delta))$-local equilibrium in a *non-concave* game, in the local regime of parameters (Lemma 1). While general Φ-regret minimization algorithms result in sub-optimal guarantees and prohibitively high computational complexity (Section 3), we show that simple online learning algorithms such as Online Gradient Descent (GD) and Optimistic Gradient (OG) achieve desirable regret guarantees in an efficient manner.

- We show that $\Phi^{\mathcal{X}}_{\mathrm{Int},\Psi}(\delta)$-regret minimization reduces to Ψ-regret minimization (Theorem 1). Notably, there are several scenarios where Ψ-regret minimization proves to be tractable e.g., when Ψ is the set of all constant strategy modifications, or settings explored in Hazan and Kale [2007], Stoltz and Lugosi [2007], Gordon et al. [2008]. We use $\Phi^{\mathcal{X}}_{\mathrm{Int}}(\delta)$ to denote the local strategy

[5] It is possible to allow the modification to select a displacement vector based on the input strategy and then project it back to the feasible set if necessary. We do not consider this general version here, as the strategy modification set becomes as general as possible and contains all δ-local strategy modifications.

modifications $\Phi^{\mathcal{X}}_{\mathrm{Int},\Psi}(\delta)$ when Ψ is the set of all constant strategy modifications. Clearly, GD or any no-external-regret learning algorithm achieves no $\Phi^{\mathcal{X}}_{\mathrm{Int}}(\delta)$-regret. Consequently, dynamics in which each player employs GD or any no-regret learning algorithm efficiently converge to an $(\varepsilon, \Phi_{\mathrm{Int}}(\delta))$-local equilibrium in the local regime. See Theorem 2.

- Unlike the $\Phi^{\mathcal{X}}_{\mathrm{Int}}(\delta)$-regret, the notion of $\Phi^{\mathcal{X}}_{\mathrm{Proj}}(\delta)$-regret is incomparable to external regret (Examples 1 and 2). However, somewhat surprisingly, via a novel analysis we show that GD and OG achieve a near-optimal, $O(\sqrt{T})$, $\Phi^{\mathcal{X}}_{\mathrm{Proj}}(\delta)$-regret guarantee without any modification (Theorem 3 and Theorem 5). Moreover, when all players employ OG, each player enjoys an improved, $O(T^{1/4})$, $\Phi^{\mathcal{X}}_{\mathrm{Proj}}(\delta)$-regret (Theorem 6), breaking the $\Omega(\sqrt{T})$ lower bound in the adversarial setting (Theorem 4).

Our results complement existing results for learning in concave games. We show that the notions of $(\varepsilon, \Phi_{\mathrm{Int}}(\delta))$-local equilibrium and $(\varepsilon, \Phi_{\mathrm{Proj}}(\delta))$-local equilibrium capture the efficient convergence guarantees enjoyed by GD, OG, and other no-regret learning dynamics in non-concave games. Interestingly, we also reveal that GD, while effective, is not universally applicable. Specifically, there exist natural local equilibrium concepts for which GD provably fails to reach a local equilibrium. See Section 6 for details.

Hardness in the Global Regime. A natural question that we have not addressed yet is whether correlation is sufficiently powerful so that our solution concept becomes tractable even in the global regime of parameters (i.e. for large δ). We provide a negative answer to this question by showing that when δ equals the diameter of our strategy set, it is NP-hard to compute an $(\varepsilon, \Phi_{\mathrm{Int}}(\delta))$-local equilibrium or an $(\varepsilon, \Phi_{\mathrm{Proj}}(\delta))$-local equilibrium, even when $\varepsilon = \Theta(1)$ and $G, L = O(\mathrm{poly}(d))$. Moreover, given black-box access to value and gradient queries, finding an $(\varepsilon, \Phi_{\mathrm{Int}}(\delta))$-local equilibrium or an $(\varepsilon, \Phi_{\mathrm{Proj}}(\delta))$-local equilibrium requires exponentially many queries in at least one of the parameters $d, G, L, 1/\varepsilon$. These results are presented as Theorem 8 and Theorem 9 in Appendix E.

1.2 Related Works

Non-Concave Games. An important special case of multi-player games are two-player zero-sum games, which are defined in terms of some function $f : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ that one of the two players, say the one choosing $x \in \mathcal{X}$, wants to minimize, while the other player, the one choosing $y \in \mathcal{Y}$, wants to maximize. Finding Nash equilibrium in such games is tractable in the *convex-concave* setting, i.e. when $f(x, y)$ is convex with respect to the minimizing player's strategy, x, and concave with respect to the maximizing player's strategy, y, but it is computationally intractable in the general *nonconvex-nonconcave* setting. Namely, a Nash equilibrium may not exist, and it is NP-hard to determine if one exists and, if so, find it. Moreover, in this case, stable limit points of gradient-based dynamics are not necessarily Nash equilibria, not even local Nash equilibria [Daskalakis and Panageas, 2018, Mazumdar et al., 2020]. There is a line of work focusing on computing Nash equilibrium under additional structure in the game. This encompasses settings where the game satisfies the (weak) Minty variational inequality [Mertikopoulos and Zhou, 2019, Diakonikolas et al., 2021, Pethick et al., 2022, Cai and Zheng, 2023], or is sufficiently close to being bilinear [Abernethy et al., 2021]. However, the study of universal solution concepts in the nonconvex-nonconcave setting is sparse. Daskalakis et al. [2021b] proved the existence and computational hardness of local Nash equilibrium. In a more recent work, [Daskalakis et al., 2023] proposes second-order algorithms with asymptotic convergence to local Nash equilibrium. Several works study sequential two-player zero-sum games, and make additional assumptions about the player who goes second. They propose equilibrium concepts such as *local minimax points* [Jin et al., 2020], *differentiable Stackelberg equilibrium* [Fiez et al., 2020], and *greedy adversarial equilibrium* [Mangoubi and Vishnoi, 2021]. Notably, local minimax points are stable limit points of Gradient-Descent-Ascent (GDA) dynamics [Jin et al., 2020, Wang et al., 2020, Fiez and Ratliff, 2021] while greedy adversarial equilibrium can be computed efficiently using second-order algorithms in the unconstrained setting [Mangoubi and Vishnoi, 2021]. In contrast to these studies, we focus on the more general case of multi-player non-concave games.

Online Learning with Non-Convex Losses A line of work has studied online learning against non-convex losses. To circumvent the computational intractability of this problem, various approaches have been pursued: some works assume a restricted set of non-convex loss functions [Gao et al., 2018], while others assume access to a sampling oracle [Maillard and Munos, 2010, Krichene et al., 2015] or access to an offline optimization oracle [Agarwal et al., 2019, Suggala and Netrapalli, 2020, Héliou et al., 2020] or a weaker notion of regret [Hazan et al., 2017, Aydore et al., 2019, Hallak et al., 2021, Guan et al., 2023]. The work most closely related to ours is [Hazan et al., 2017]. The authors propose a notion of w*-smoothed local regret* against non-convex losses, and they also define a local equilibrium concept for non-concave games. They use the idea of *smoothing* to average the loss functions in the previous w iterations and design algorithms with optimal w-smoothed local regret. The concept of regret they introduce suggests a local equilibrium concept. However, their local equilibrium concept is non-standard in that its local stability is not with respect to a distribution over strategy profiles sampled by this equilibrium concept. Moreover, the path to attaining this local equilibrium through decentralized learning dynamics remains unclear. The algorithms provided in [Hazan et al., 2017, Guan et al., 2023] require that every agent i experiences (over several rounds) the average utility function of the previous w iterates, denoted as $F_{i,w}^t := \frac{1}{w}\sum_{\ell=0}^{w-1} u_i^{t-\ell}(\cdot, x_{-i}^{t-\ell})$. Implementing this imposes significant coordination burden on the agents. In contrast, we focus on a natural concept of $(\varepsilon, \Phi(\delta))$-local equilibrium, which is incomparable to that of Hazan et al. [2017], and we also show that efficient convergence to this concept is achieved via decentralized gradient-based learning dynamics.

Φ-regret and Φ-equilibrium The concept of Φ-regret and the associated Φ-equilibrium is introduced by Greenwald and Jafari [2003] and has been broadly investigated in the context of concave games [Greenwald and Jafari, 2003, Stoltz and Lugosi, 2007, Gordon et al., 2008, Rakhlin et al., 2011, Piliouras et al., 2022, Bernasconi et al., 2023] and extensive-form games [Von Stengel and Forges, 2008, Morrill et al., 2021a,b, Farina et al., 2022b, Bai et al., 2022, Song et al., 2022, Anagnostides et al., 2023]. The two types of $(\varepsilon, \Phi(\delta))$-local equilibrium we explore can be regarded as special cases of Φ-equilibrium in non-concave games, wherein all strategy modifications are localized. To the best of our knowledge, no one has yet focused on this unique setting, and no efficient algorithm is known for minimizing such Φ-regret or computing the corresponding Φ-equilibrium.

2 Preliminaries

A ball of radius $r > 0$ centered at $x \in \mathbb{R}^d$ is denoted by $B_d(x, r) := \{x' \in \mathbb{R}^d : \|x - x'\| \le r\}$. We use $\|\cdot\|$ for L_2 norm throughout. We also write $B_d(\delta)$ for a ball centered at the origin with radius δ. For $a \in \mathbb{R}$, we use $[a]^+$ to denote $\max\{0, a\}$. We denote $D_{\mathcal{X}}$ the diameter of a set $\mathcal{X}$.

Differentiable / Smooth Games. An n-player *differentiable game* has a set of n players $[n] := \{1, 2, \ldots, n\}$. Each player $i \in [n]$ has a nonempty convex and compact strategy set $\mathcal{X}_i \subseteq \mathbb{R}^{d_i}$. For a joint strategy profile $x = (x_i, x_{-i}) \in \prod_{j=1}^n \mathcal{X}_j$, the reward of player i is determined by a utility function $u_i : \prod_{j=1}^n \mathcal{X}_j \to \mathbb{R}$ whose gradient with respect to x_i is continuous. We denote by $d = \sum_{i=1}^n d_i$ the dimensionality of the game and assume $\max_{i \in [n]}\{D_{\mathcal{X}_i}\} \le D$. A *smooth game* is a differentiable game whose utility functions further satisfy the following assumption.

Assumption 1 (Smooth Games)**.** *The utility function $u_i(x_i, x_{-i})$ for any player $i \in [n]$ satisfies:*

1. *(G-Lipschitzness) $\|\nabla_{x_i} u_i(x)\| \le G$ for all i and $x \in \prod_{j=1}^n \mathcal{X}_j$;*
2. *(L-smoothness) there exists $L_i > 0$ such that $\|\nabla_{x_i} u_i(x_i, x_{-i}) - \nabla_{x_i} u_i(x_i', x_{-i})\| \le L_i \|x_i - x_i'\|$ for all $x_i, x_i' \in \mathcal{X}_i$ and $x_{-i} \in \Pi_{j \ne i} \mathcal{X}_j$. We denote $L = \max_i L_i$ as the smoothness of the game.*

Crucially, we make no assumption on the concavity of $u_i(x_i, x_{-i})$.

Local Nash Equilibrium. For $\varepsilon, \delta > 0$, an (ε, δ)-local Nash equilibrium is a strategy profile in which no player can increase their own utility by more than ε via a deviation bounded by δ. The formal definition is as follows.

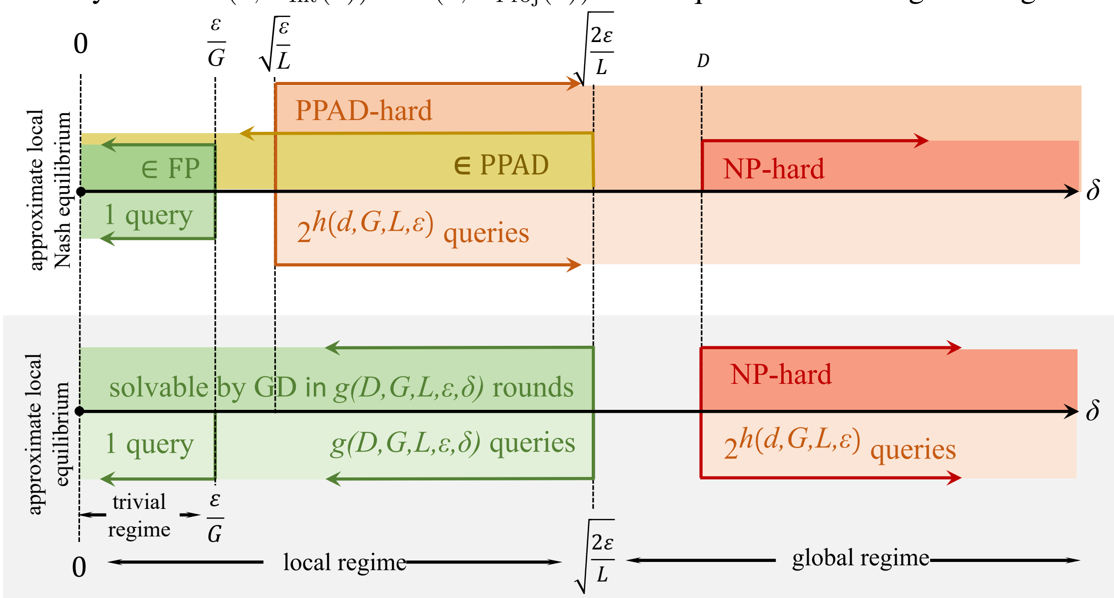
Definition 1 ((ε, δ)-Local Nash Equilibrium [Daskalakis et al., 2021b, Daskalakis, 2022])**.** *In a smooth game, for some $\varepsilon, \delta > 0$, a strategy profile $x^* \in \prod_{j=1}^n \mathcal{X}_j$ is an (ε, δ)-*local Nash equilibrium *if and only if for every player $i \in [n]$, $u_i(x_i, x_{-i}^*) \le u_i(x^*) + \varepsilon$, for every $x_i \in B_{d_i}(x_i^*, \delta) \cap \mathcal{X}_i$.*

For large enough δ, Definition 1 captures ε-global Nash equilibrium as well. The notion of (ε, δ)-local Nash equilibrium transitions from being ε-approximate local Nash equilibrium to ε-approximate Nash equilibrium as δ ranges from small to large. The complexity of computing of an (ε, δ)-local Nash equilibrium is characterized by δ as follows (see Figure 2 for a summary).

- **Trivial Regime.** When $\delta < \varepsilon / G$, then every point $x \in \prod_{j=1}^n \mathcal{X}_j$ is an (ε, δ)-local Nash equilibrium since for any player $i \in [n]$, it holds that $\|u_i(x) - u_i(x_i', x_{-i})\| \le \varepsilon$ for every $x_i \in B_{d_i}(x_i, \delta)$.
- **Local Regime.** When $\delta \le \sqrt{2\varepsilon / L}$, an (ε, δ)-local Nash equilibrium always exists. However, finding an (ε, δ)-local Nash equilibrium is PPAD-hard for any $\delta \ge \sqrt{\varepsilon / L}$, even when $1/\varepsilon, G, L = O(\mathrm{poly}(d))$ [Daskalakis et al., 2021b]. Our main focus in this paper is this local regime.

- **Global Regime.** When $\delta \geq D$, then (ε, δ)-local Nash equilibrium becomes a standard ε-Nash equilibrium and is NP-hard to find even if $\varepsilon = \Theta(1)$ and $G, L = O(\text{poly}(d))$ [Daskalakis et al., 2021b].

Figure 2: Summary of results and comparison between the complexity of computing an (ε, δ)-local Nash equilibrium and an $(\varepsilon, \Phi(\delta))$-local equilibrium (for $\Phi_{\text{Int}}(\delta)$ and $\Phi_{\text{Proj}}(\delta)$) in a d-dimensional, G-Lispchitz, and L-smooth game where each player's strategy space is bounded by D. We assume $\varepsilon \leq G^2/L$ so that the trivial regime is a subset of the local regime. The function h in the query lower bound is defined as $h(d, G, L, \varepsilon) = (\min(d, G/\varepsilon, L/\varepsilon))^p$ for some universal constant $p > 0$. The function g is defined as $g(D, G, L, \varepsilon, \delta) = \text{poly}(D, G, L, \delta, \frac{1}{(2\varepsilon - \delta^2 L)})$. PPAD-hard problems are widely believed to be computationally intractable. Our main results are (1) efficient algorithms for computing both $(\varepsilon, \Phi_{\text{Int}}(\delta))$ and $(\varepsilon, \Phi_{\text{Proj}}(\delta))$-local equilibrium in the local regime and (2) intractability for both $(\varepsilon, \Phi_{\text{Int}}(\delta))$ and $(\varepsilon, \Phi_{\text{Proj}}(\delta))$-local equilibrium in the global regime.



3 $(\varepsilon, \Phi(\delta))$-local equilibrium and Φ-regret

In this section, we introduce the concept of $(\varepsilon, \Phi(\delta))$-local equilibrium and explore its relationship with online learning and Φ-regret minimization. We first define δ-*local* strategy modification set.

Definition 2 (δ-local strategy modification)**.** *For each agent i, we call a set of strategy modifications $\Phi^{\mathcal{X}_i}$* δ-local *if for all $x \in \mathcal{X}_i$ and $\phi_i \in \Phi^{\mathcal{X}_i}$, $\|\phi_i(x) - x\| \leq \delta$. We use notation $\Phi^{\mathcal{X}_i}(\delta)$ to denote a δ-local strategy modification set for agent i. We also use $\Phi(\delta) = \Pi_{i=1}^n \Phi^{\mathcal{X}_i}(\delta)$ to denote a profile of δ-local strategy modification sets.*

We define $(\varepsilon, \Phi(\delta))$-local equilibrium of a differentiable game as a distribution over joint strategy profiles such that no player i can increase their expected utility by more than ε using any strategy modification in $\Phi^{\mathcal{X}_i}(\delta)$.

Definition 3 (($(\varepsilon, \Phi(\delta))$-local equilibrium). *In a differentiable game, a distribution σ over joint strategy profiles $\Pi_{i=1}^{n}\mathcal{X}_i$ is an $(\varepsilon, \Phi(\delta))$-local equilibrium for some $\varepsilon > 0$ and a profile of δ-local strategy modification sets $\Phi(\delta)$ if and only if for all player $i \in [n]$,*

$$\max_{\phi_i \in \Phi^{\mathcal{X}_i}(\delta)} \mathbb{E}_{x\sim\sigma}[u_i(\phi_i(x_i), x_{-i})] \leq \mathbb{E}_{x\sim\sigma}[u_i(x)] + \varepsilon.$$

By the definition of local modification, any (ε, δ)-local Nash equilibrium is also an $(\varepsilon, \Phi(\delta))$-local equilibrium for any set $\Phi(\delta)$. Thus in the local regime where $\delta \leq \sqrt{2\varepsilon/L}$, the existence of $(\varepsilon, \Phi(\delta))$-local equilibrium for any set of δ-local strategy modifications $\Phi(\delta)$ follows from the existence of (ε, δ)-local Nash equilibrium, which is established in [Daskalakis et al., 2021b]. $(\varepsilon, \Phi(\delta))$-local equilibrium is closely related to the notion of Φ-regret minimization in online learning. Below, we first present some background on online learning and Φ-regret.

Online Learning and Φ-Regret. We consider the standard online learning setting: at each day $t \in [T]$, the learner chooses an action x^t from a nonempty convex compact set $\mathcal{X} \subseteq \mathbb{R}^m$ and the adversary chooses a possibly non-convex loss function $f^t : \mathcal{X} \to \mathbb{R}$, then the learner suffers a loss $f^t(x^t)$ and receives gradient feedback $\nabla f^t(x^t)$. We make the same assumptions on $\{f^t\}_{t\in[T]}$ as in Assumption 1 that each f^t is G-Lipschitz, and L-smooth. The classic goal of an online learning algorithm is to minimize the *external regret* defined as $\mathrm{Reg}^T := \max_{x\in\mathcal{X}} \sum_{t=1}^{T}(f^t(x^t) - f^t(x))$. An algorithm is called *no-regret* if its external regret is sublinear in T. The notion of Φ-regret generalizes external regret by allowing more general strategy modifications.

Definition 4 (Φ-regret). *Let Φ be a set of strategy modification functions $\{\phi : \mathcal{X} \to \mathcal{X}\}$. For $T \geq 1$, the Φ-regret of an online learning algorithm is*

$$\mathrm{Reg}_{\Phi}^{T} := \max_{\phi\in\Phi} \sum_{t=1}^{T} \left(f^t(x^t) - f^t(\phi(x^t))\right).$$

An algorithm has no Φ-regret *if its Φ-regret is sublinear in T.*

Many classic notions of regret can be interpreted as Φ-regret. For example, the external regret is Φ_{ext}-regret where Φ_{ext} contains all constant strategy modifications $\phi_{x^*}(x) = x^*$ for all $x^* \in \mathcal{X}$. The *swap regret* on simplex Δ^m is Φ_{swap}-regret where Φ_{swap} contains all linear transformations $\phi : \Delta^m \to \Delta^m$.

The main result of the section is a reduction from $(\varepsilon, \Phi(\delta))$-local equilibrium computation for *non-concave* smooth games in the local regime, to $\Phi^{\mathcal{X}_i}(\delta)$-regret minimization against *convex* losses. The key observation here is that the L-smoothness of the utility function permits the approximation of a non-concave function with a linear function within a local area bounded by δ. This approximation yields an error of at most $\frac{\delta^2 L}{2}$, which is less than ε in the local regime. We defer the proof to Appendix B.

Lemma 1 (No $\Phi(\delta)$-Regret to $(\varepsilon, \Phi(\delta))$-Local Equilibrium). *For any $T \geq 1$ and $\delta > 0$, let $\mathcal{A}$ be an algorithm that guarantees to achieve no more than $\mathrm{Reg}^T_{\Phi^{\mathcal{X}_i}(\delta)}$ $\Phi^{\mathcal{X}_i}(\delta)$-regret for* convex *loss functions for each agent $i \in [n]$. Then*

1. *The $\Phi^{\mathcal{X}_i}(\delta)$-regret of $\mathcal{A}$ for* non-convex *and L-smooth loss functions is at most $\mathrm{Reg}^T_{\Phi^{\mathcal{X}_i}(\delta)} + \frac{\delta^2 LT}{2}$ for each agent i.*

2. *When every agent employs $\mathcal{A}$ in a non-concave L-smooth game, their empirical distribution of the joint strategies played converges to a $\left(\max_{i\in[n]}\{\mathrm{Reg}^T_{\Phi^{\mathcal{X}_i}(\delta)}\}/T + \frac{\delta^2 L}{2}, \Phi(\delta)\right)$-local equilibrium.*

Computing Local Equilibrium via Generic Φ-Regret Minimization. By Lemma 1, it suffices to design no Φ-regret algorithms against convex losses for efficient equilibrium computation. Although Φ-regret minimization is extensively studied [Greenwald and Jafari, 2003, Hazan and Kale, 2007, Stoltz and Lugosi, 2007, Gordon et al., 2008, Dagan et al., 2023, Peng and Rubinstein, 2023], to our knowledge, all of these algorithms, when applied to compute a $(\varepsilon, \Phi(\delta))$-Local Equilibrium for a general δ-local strategy modification set $\Phi(\delta)$ (using Lemma 1), require running time exponential in either $1/\varepsilon$ or the dimension d. In the following sections, we show that for several natural choices of $\Phi(\delta)$, $(\varepsilon, \Phi(\delta))$-local equilibrium can be computed efficiently, i.e. polynomial in $1/\varepsilon$ and d, using simple algorithms.

4 Interpolation-Based Local Strategy Modifications and Local Equilibria

We introduce a natural set of local strategy modifications and the corresponding local equilibrium notion. Given any set of (possibly non-local) strategy modifications $\Psi = \{\psi : \mathcal{X} \to \mathcal{X}\}$, we define a set of *local* strategy modifications as follows: for $\delta \le D_{\mathcal{X}}$ and $\lambda \in [0,1]$, each strategy modification $\phi_{\lambda,\psi}$ interpolates the input strategy x with the modified strategy $\psi(x)$: formally,

$$\Phi^{\mathcal{X}}_{\mathrm{Int},\Psi}(\delta) := \left\{\phi_{\lambda,\psi}(x) := (1-\lambda)x + \lambda\psi(x) : \psi \in \Psi, \lambda \le \delta/D_{\mathcal{X}}\right\}.$$

Note that for any $\psi \in \Psi$ and $\lambda \le \frac{\delta}{D_{\mathcal{X}}}$, we have $\|\phi_{\lambda,\psi}(x) - x\| = \lambda\|x - \psi(x)\| \le \delta$, respecting the locality constraint. The induced $\Phi^{\mathcal{X}}_{\mathrm{Int},\Psi}(\delta)$-regret can be written as $\mathrm{Reg}^T_{\mathrm{Int},\Psi,\delta} := \max_{\psi\in\Psi,\lambda\le\frac{\delta}{D_{\mathcal{X}}}} \sum_{t=1}^T \left(f^t(x^t) - f^t((1-\lambda)x^t + \lambda\psi(x^t))\right)$. We now define the corresponding $(\varepsilon, \Phi_{\mathrm{Int},\Psi}(\delta))$-local equilibrium.

Definition 5. *Define $\Phi_{\mathrm{Int},\Psi}(\delta) = \Pi_{j=1}^n \Phi^{\mathcal{X}_j}_{\mathrm{Int},\Psi_j}(\delta)$. In a differentiable game, a distribution σ over strategy profiles is a* $(\varepsilon, \Phi_{\mathrm{Int},\Psi}(\delta))$-local equilibrium *if and only if for all player $i \in [n]$,*

$$\max_{\psi\in\Psi_i,\lambda\le\delta/D_{\mathcal{X}_i}} \mathbb{E}_{x\sim\sigma}[u_i((1-\lambda)x_i + \lambda\psi(x_i), x_{-i})] \le \mathbb{E}_{x\sim\sigma}[u_i(x)] + \varepsilon.$$

Intuitively speaking, when a correlation device recommends strategies to players according to an $(\varepsilon, \Phi_{\mathrm{Int},\Psi}(\delta))$-local equilibrium, no player can increase their utility by more than ε through a local deviation by interpolating with a (possibly global) strategy modification $\psi \in \Psi$. The richness of Ψ determines the incentive guarantee provided by an $(\varepsilon, \Phi_{\mathrm{Int},\Psi}(\delta))$-local equilibrium as well as its computational complexity. When we choose Ψ to be the set of all possible strategy modifications, the corresponding notion of local equilibrium—limiting the gain of a player by interpolating with any strategy—resembles that of a *correlated equilibrium*.

Computation of $(\varepsilon, \Phi_{\mathrm{Int},\Psi}(\delta))$-Local Equilibrium. By Lemma 1, we know computing an $(\varepsilon, \Phi_{\mathrm{Int},\Psi}(\delta))$-local equilibrium reduces to minimizing $\Phi^{\mathcal{X}}_{\mathrm{Int},\Psi}(\delta)$-regret against convex loss functions. We show that minimizing $\Phi^{\mathcal{X}}_{\mathrm{Int},\Psi}(\delta)$-regret against convex loss functions further reduces to Ψ-regret minimization against linear loss functions.

Theorem 1. *Let $\mathcal{A}$ be an algorithm with Ψ-regret $\mathrm{Reg}^T_\Psi(G, D_{\mathcal{X}})$ for linear and G-Lipschitz loss functions over $\mathcal{X}$. Then, for any $\delta > 0$, the $\Phi^{\mathcal{X}}_{\mathrm{Int},\Psi}(\delta)$-regret of $\mathcal{A}$ for convex and G-Lipschitz loss functions over $\mathcal{X}$ is at most $\frac{\delta}{D_{\mathcal{X}}} \cdot [\mathrm{Reg}^T_\Psi(G, D_{\mathcal{X}})]^+$.*

Proof. By definition and convexity of f^t, we get

$$\max_{\phi \in \Phi^{\mathcal{X}}_{\mathrm{Int},\Psi}(\delta)} \sum_{t=1}^T f^t(x^t) - f^t(\phi(x^t)) = \max_{\psi \in \Psi, \lambda \le \frac{\delta}{D_{\mathcal{X}}}} \sum_{t=1}^T f^t(x^t) - f^t((1-\lambda)x^t + \lambda \psi(x^t))$$
$$\le \frac{\delta}{D_{\mathcal{X}}} \left[\max_{\psi \in \Psi} \sum_{t=1}^T \left\langle \nabla f^t(x^t), x^t - \psi(x^t) \right\rangle \right]^+ .$$

□

Note that when f^t is linear, the reduction is without loss. Thus, any worst-case $\Omega(r(T))$-lower bound for Ψ-regret implies a $\Omega(\frac{\delta}{D_{\mathcal{X}}} \cdot r(T))$ lower bound for $\Phi_{\mathrm{Int},\Psi}(\delta)$-regret. Moreover, for any set Ψ that admits efficient Ψ-regret minimization algorithms such as swap transformations over the simplex and more generally any set such that (i) all modifications in the set can be represented as linear transformations in some finite-dimensional space and (ii) fixed point computation can be carried out efficiently for any linear transformations [Gordon et al., 2008], we also get an efficient algorithm for computing an $(\varepsilon, \Phi_{\mathrm{Int},\Psi}(\delta))$-local equilibrium in the local regime.

CCE-like Instantiation In the special case where Ψ contains only *constant* strategy modifications (i.e. $\psi(x) = x^*$ for all x), we get a coarse correlated equilibrium (CCE)-like instantiation of local equilibrium, which limits the gain by interpolating with any *fixed* strategy. We denote the resulting set of local strategy modification simply as $\Phi^{\mathcal{X}}_{\mathrm{Int}}$. We can apply any no-external regret algorithm for efficient $\Phi^{\mathcal{X}}_{\mathrm{Int}}$-regret minimization and computation of $(\varepsilon, \Phi_{\mathrm{Int}}(\delta))$-local equilibrium in the local regime as summarized in Theorem 2.

Theorem 2. *For the Online Gradient Descent algorithm (GD) [Zinkevich, 2003] with step size $\eta = \frac{D_{\mathcal{X}}}{G\sqrt{T}}$, its $\Phi^{\mathcal{X}}_{\mathrm{Int}}(\delta)$-regret is at most $2\delta G\sqrt{T}$. Furthermore, for any $\delta > 0$ and any $\varepsilon > \frac{\delta^2 L}{2}$, when all players employ the GD algorithm in a smooth game, their empirical distribution of played strategy profiles converges to an $(\varepsilon, \Phi_{\mathrm{Int}}(\delta))$-local equilibrium in $\frac{16\delta^2 G^2}{(2\varepsilon - \delta^2 L)^2} = O(1/\varepsilon^2)$ iterations.*

The above $\Phi^{\mathcal{X}}_{\mathrm{Int}}(\delta)$-regret bound of $O(\sqrt{T})$ is derived for the adversarial setting. In the game setting, where each player employs the same algorithm, players may have substantially lower external regret [Syrgkanis et al., 2015, Chen and Peng, 2020, Daskalakis et al., 2021a, Anagnostides et al., 2022a,b, Farina et al., 2022a] but we need a slightly stronger smoothness assumption than Assumption 1. This assumption is naturally satisfied by finite normal-form games and is also made for results about concave games [Farina et al., 2022a].

Assumption 2. *For any player $i \in [n]$, the utility $u_i(x)$ satisfies $\|\nabla_{x_i} u_i(x) - \nabla_{x_i} u_i(x')\| \le L\|x - x'\|$ for all $x, x' \in \mathcal{X}$.*

Using Assumption 2 and Lemma 1, the no-regret learning dynamics of [Farina et al., 2022a] that guarantees $O(\log T)$ individual external regret in concave games can be applied to smooth non-concave games so that the individual $\Phi^{\mathcal{X}}_{\text{Int}}(\delta)$-regret of each player is at most $O(\log T) + \frac{\delta^2 LT}{2}$. This gives an algorithm with faster $\tilde{O}(1/\varepsilon)$ convergence to an $(\varepsilon, \Phi_{\text{Int}}(\delta))$-local equilibrium than GD.

5 Projection-Based Local Strategy Modifications and Local Equilibria

In this section, we study a set of local strategy modifications based on projection. Specifically, the set $\Phi^{\mathcal{X}}_{\text{Proj}}(\delta)$ encompasses all deviations that essentially add a fixed displacement vector v to the input strategy and project back to the feasible set:

$$\Phi^{\mathcal{X}}_{\text{Proj}}(\delta) := \{\phi_{\text{Proj},v}(x) = \Pi_{\mathcal{X}}[x - v] : v \in B_d(\delta)\}.$$

It is clear that $\|\phi_v(x) - x\| \le \|v\| \le \delta$. The induced $\Phi^{\mathcal{X}}_{\text{Proj}}(\delta)$-regret is

$$\text{Reg}^T_{\text{Proj},\delta} := \max_{v \in B_d(\delta)} \sum_{t=1}^{T} \left(f^t(x^t) - f^t(\Pi_{\mathcal{X}}[x^t - v])\right).$$

The set $\Phi^{\mathcal{X}}_{\text{Proj}}(\delta)$ also induce a notion of $(\varepsilon, \Phi(\delta))$-local equilibrium. As the displacement vector remains constant for each modification in $\Phi^{\mathcal{X}}_{\text{Proj}}(\delta)$, we interpret the corresponding local equilibrium as a notion similar to a CCE.

Definition 6. *Define $\Phi_{\text{Proj}}(\delta) = \Pi_{j=1}^{n} \Phi^{\mathcal{X}_j}_{\text{Proj}}(\delta)$. In a differentiable game, a distribution σ over joint strategy profiles is an $(\varepsilon, \Phi_{\text{Proj}}(\delta))$-*local equilibrium *if and only if for all player $i \in [n]$,*

$$\max_{v \in B_d(\delta)} \mathbb{E}_{x \sim \sigma}[u_i(\Pi_{\mathcal{X}}[x_i - v], x_{-i})] \le \mathbb{E}_{x \sim \sigma}[u_i(x)] + \varepsilon.$$

By definition, when a correlation device recommends a strategy to each player according to an $(\varepsilon, \Phi_{\text{Proj}}(\delta))$-local equilibrium, no player can gain more than ε by a fixed-direction deviation bounded by δ. Unlike the $\Phi^{\mathcal{X}}_{\text{Int}}(\delta)$-regret, we can not directly reduce $\Phi^{\mathcal{X}}_{\text{Proj}}(\delta)$-regret minimization to external regret minimization. Below, we first illustrate that external regret and $\Phi^{\mathcal{X}}_{\text{Proj}}(\delta)$-regret are incomparable, and why a reduction similar to the one we show in Section 4 is unlikely to exist. Despite this, surprisingly, we show that classical algorithms like Online Gradient Descent (GD) and Optimistic Gradient (OG), known for minimizing external regret, also attain near-optimal $\Phi^{\mathcal{X}}_{\text{Proj}}(\delta)$-regret. Missing proofs of this section are in Appendix C.

Difference between external regret and $\Phi^{\mathcal{X}}_{\mathrm{Proj}}(\delta)$-regret In the following two examples, we show that $\Phi^{\mathcal{X}}_{\mathrm{Proj}}(\delta)$-regret is incomparable with external regret for convex loss functions . A sequence of actions may suffer high Reg^T but low $\mathrm{Reg}^T_{\mathrm{Proj},\delta}$ (Example 1), and vise versa (Example 2).

Example 1. *Let $f^1(x) = f^2(x) = |x|$ for $x \in \mathcal{X} = [-1, 1]$. Then the $\Phi^{\mathcal{X}}_{\mathrm{Proj}}(\delta)$-regret of the sequence $\{x^1 = \frac{1}{2}, x^2 = -\frac{1}{2}\}$ for any $\delta \in (0, \frac{1}{2})$ is 0. However, the external regret of the same sequence is 1. By repeating the construction for $\frac{T}{2}$ times, we conclude that there exists a sequence of actions with $\mathrm{Reg}^T_{\mathrm{Proj},\delta} = 0$ and $\mathrm{Reg}^T = \frac{T}{2}$ for all $T \geq 2$.*

Example 2. *Let $f^1(x) = -2x$ and $f^2(x) = x$ for $x \in \mathcal{X} = [-1, 1]$. Then the $\Phi^{\mathcal{X}}_{\mathrm{Proj}}(\delta)$-regret of the sequence $\{x^1 = \frac{1}{2}, x^2 = 0\}$ for any $\delta \in (0, \frac{1}{2})$ is δ. However, the external regret of the same sequence is 0. By repeating the construction for $\frac{T}{2}$ times, we conclude that there exists a sequence of actions with $\mathrm{Reg}^T_{\mathrm{Proj},\delta} = \frac{\delta T}{2}$ and $\mathrm{Reg}^T = 0$ for all $T \geq 2$.*

At a high level, the external regret competes against a fixed action, whereas $\Phi^{\mathcal{X}}_{\mathrm{Proj}}(\delta)$-regret is more akin to the notion of *dynamic regret*, competing with a sequence of varying actions. When the environment is stationary, i.e., $f^t = f$ (Example 1), a sequence of actions that are far away from the global minimum must suffer high regret, but may produce low $\Phi^{\mathcal{X}}_{\mathrm{Proj}}(\delta)$-regret since the change to the cumulative loss caused by a fixed-direction deviation could be neutralized across different actions in the sequence. In contrast, in a non-stationary (dynamic) environment (Example 2), every fixed action performs poorly, and a sequence of actions could suffer low regret against a fixed action but the $\Phi^{\mathcal{X}}_{\mathrm{Proj}}(\delta)$-regret that competes with a fixed-direction deviation could be large. The fact that small external regret does not necessarily equate to small $\Phi^{\mathcal{X}}_{\mathrm{Proj}}(\delta)$-regret is in sharp contrast to the behavior of the $\Phi^{\mathcal{X}}_{\mathrm{Int}}(\delta)$-regret. Nevertheless, despite these differences between the two notions of regret as shown above, they are *compatible* for convex loss functions: our main results in this section provide algorithms that minimize external regret and $\Phi^{\mathcal{X}}_{\mathrm{Proj}}(\delta)$-regret simultaneously.

5.1 $\Phi^{\mathcal{X}}_{\mathrm{Proj}}(\delta)$-Regret minimization in the adversarial setting

In this section, we show that the classic Online Gradient Descent (GD) algorithm enjoys an $O(G\sqrt{\delta D_{\mathcal{X}} T})$ $\Phi^{\mathcal{X}}_{\mathrm{Proj}}(\delta)$-regret despite the difference between the external regret and $\Phi^{\mathcal{X}}_{\mathrm{Proj}}(\delta)$-regret. First, let us recall the update rule of GD: given initial point $x^1 \in \mathcal{X}$ and step size $\eta > 0$, GD updates in each iteration t:

$$x^{t+1} = \Pi_{\mathcal{X}}[x - \eta \nabla f^t(x^t)]. \tag{GD}$$

The key step in our analysis for GD is simple but novel and general (See Appendix C). We can extend the analysis to many other algorithms such as Optimistic Gradient (OG) in Section 5.2.

Theorem 3. *Let $\delta > 0$ and $T \in \mathbb{N}$. For convex and G-Lipschitz loss functions $\{f^t : \mathcal{X} \to \mathbb{R}\}_{t\in[T]}$, the $\Phi^{\mathcal{X}}_{\mathrm{Proj}}(\delta)$-regret of (GD) with step size $\eta > 0$ is*

$$\mathrm{Reg}^T_{\mathrm{Proj},\delta} \leq \frac{\delta^2}{2\eta} + \frac{\eta}{2}G^2T + \frac{\delta D_{\mathcal{X}}}{\eta}.$$

We can choose η optimally as $\frac{\sqrt{\delta(\delta+D_{\mathcal{X}})}}{G\sqrt{T}}$ and attain $\mathrm{Reg}^T_{\mathrm{Proj},\delta} \le 2G\sqrt{\delta(\delta+D_{\mathcal{X}})T}$. For any $\delta > 0$ and any $\varepsilon > \frac{\delta^2 L}{2}$, when all player employ GD in a smooth game, their empirical distribution of played strategy profiles converges to an $(\varepsilon, \Phi_{\mathrm{Proj}}(\delta))$-local equilibrium in $O(\frac{\delta D G^2}{(2\varepsilon-\delta^2 L)^2}) = O(1/\varepsilon^2)$ iterations.

Remark 1. *Note that $\Phi^{\mathcal{X}}_{\mathrm{Proj}}(\delta)$-regret can also be viewed as the* dynamic regret *[Zinkevich, 2003] with changing comparators $\{p^t := \Pi_{\mathcal{X}}[x - v]\}$. However, we remark that our analysis does not follow from standard $O(\frac{(1+P_T)}{\eta} + \eta T)$ dynamic regret bound of GD [Zinkevich, 2003] since P_T, defined as $\sum_{t=2}^T \|p^t - p^{t-1}\|$, can be $\Omega(\eta T)$.*

Proof. Let us denote $v \in B_d(\delta)$ a fixed deviation and define $p^t = \Pi_{\mathcal{X}}[x^t - v]$. By standard analysis of GD [Zinkevich, 2003] (see also the proof of [Bubeck et al., 2015, Theorem 3.2]), we have

$$\begin{aligned}
\sum_{t=1}^T \left(f^t(x^t) - f^t(p^t)\right) &\le \sum_{t=1}^T \frac{1}{2\eta}\left(\left\|x^t - p^t\right\|^2 - \left\|x^{t+1} - p^t\right\|^2 + \eta^2\left\|\nabla f^t(x^t)\right\|^2\right) \\
&\le \sum_{t=1}^{T-1} \frac{1}{2\eta}\left(\left\|x^{t+1} - p^{t+1}\right\|^2 - \left\|x^{t+1} - p^t\right\|^2\right) + \frac{\delta^2}{2\eta} + \frac{\eta}{2}G^2 T,
\end{aligned}$$

where the last step uses $\|x^1 - p^1\| \le \delta$ and $\|\nabla f^t(x^t)\| \le G$. Here the terms $\|x^{t+1} - p^{t+1}\|^2 - \|x^{t+1} - p^t\|^2$ do not telescope, and we further relax them in the following key step.

Key Step: We relax the first term as:

$$\begin{aligned}
\left\|x^{t+1} - p^{t+1}\right\|^2 - \left\|x^{t+1} - p^t\right\|^2 &= \left\langle p^t - p^{t+1}, 2x^{t+1} - p^t - p^{t+1}\right\rangle \\
&= \left\langle p^t - p^{t+1}, 2x^{t+1} - 2p^{t+1}\right\rangle - \left\|p^t - p^{t+1}\right\|^2 \\
&= 2\left\langle p^t - p^{t+1}, v\right\rangle + 2\left\langle p^t - p^{t+1}, x^{t+1} - v - p^{t+1}\right\rangle - \left\|p^t - p^{t+1}\right\|^2 \\
&\le 2\left\langle p^t - p^{t+1}, v\right\rangle - \left\|p^t - p^{t+1}\right\|^2,
\end{aligned}$$

where in the last inequality we use the fact that p^{t+1} is the projection of $x^{t+1} - v$ onto $\mathcal{X}$ and p^t is in $\mathcal{X}$. Now we get a telescoping term $2\langle p^t - p^{t+1}, u\rangle$ and a negative term $-\|p^t - p^{t+1}\|^2$. The negative term is useful for improving the regret analysis in the game setting, but we ignore it for now. Combining the two inequalities above, we have

$$\begin{aligned}
\sum_{t=1}^T \left(f^t(x^t) - f^t(p^t)\right) &\le \frac{\delta^2}{2\eta} + \frac{\eta}{2}G^2 T + \frac{1}{\eta}\sum_{t=1}^{T-1}\left\langle p^t - p^{t+1}, v\right\rangle \\
&= \frac{\delta^2}{2\eta} + \frac{\eta}{2}G^2 T + \frac{1}{\eta}\left\langle p^1 - p^T, v\right\rangle \le \frac{\delta^2}{2\eta} + \frac{\eta}{2}G^2 T + \frac{\delta D_{\mathcal{X}}}{\eta}.
\end{aligned}$$

Since the above holds for any v with $\|v\| \le \delta$, it also upper bounds $\mathrm{Reg}^T_{\mathrm{Proj},\delta}$. □

Lower bounds for $\Phi^{\mathcal{X}}_{\mathrm{Proj}}(\delta)$-regret We complement our upper bound with two lower bounds for $\Phi^{\mathcal{X}}_{\mathrm{Proj}}(\delta)$-regret minimization. The first one is an $\Omega(\delta G\sqrt{T})$ lower bound for any online learning algorithms against linear loss functions. The proof of Theorem 4 is postponed to Appendix C. Note that linear functions are 0-smooth, so the same lower bound holds for G-Lipschitz and L-smooth convex loss functions.

Theorem 4 (Lower bound for $\Phi^{\mathcal{X}}_{\mathrm{Proj}}(\delta)$-regret against convex losses)**.** *For any $T \ge 1$, $D_{\mathcal{X}} > 0$, $0 < \delta \le D_{\mathcal{X}}$, and $G \ge 0$, there exists a distribution $\mathcal{D}$ on G-Lipschitz linear loss functions $f^1, \dots, f^T$ over $\mathcal{X} = [-D_{\mathcal{X}}, D_{\mathcal{X}}]$ such that for any online algorithm, its $\Phi^{\mathcal{X}}_{\mathrm{Proj}}(\delta)$-regret on the loss sequence satisfies*

$$\mathbb{E}_{\mathcal{D}}\left[\mathrm{Reg}^T_{\mathrm{Proj},\delta}\right] = \Omega(\delta G\sqrt{T}).$$

Remark 2. *A keen reader may notice that the $\Omega(G\delta\sqrt{T})$ lower bound in Theorem 4 does not match the $O(G\sqrt{\delta D_{\mathcal{X}} T})$ upper bound in Theorem 3, especially when $D_{\mathcal{X}} \gg \delta$. A natural question is:* which of them is tight? *We conjecture that the lower bound is tight. In fact, for the special case where the feasible set $\mathcal{X}$ is a* box*, we obtain a $D_{\mathcal{X}}$-independent bound $O(d^{\frac{1}{4}}G\delta\sqrt{T})$ using a modified version of GD, which is tight when $d = 1$. See Appendix F for a detailed discussion.*

This lower bound suggests that GD achieves near-optimal $\Phi^{\mathcal{X}}_{\mathrm{Proj}}(\delta)$-regret for convex losses. For L-smooth *non-convex* loss functions, we provide another $\Omega(\delta^2 LT)$ lower bound for algorithms that satisfy the linear span assumption. The *linear span* assumption states that the algorithm produces $x^{t+1} \in \{\Pi_{\mathcal{X}}[\sum_{i\in[t]} a_i \cdot x^i + b_i \cdot \nabla f^i(x^i)] : a_i, b_i \in \mathbb{R}, \forall i \in [t]\}$ as essentially the linear combination of the previous iterates and their gradients. Many online algorithms such as online gradient descent and optimistic gradient satisfy the linear span assumption. Combining with Lemma 1, this lower bound suggests that GD attains nearly optimal $\Phi^{\mathcal{X}}_{\mathrm{Proj}}(\delta)$-regret, even in the non-convex setting, among a natural family of gradient-based algorithms.

Proposition 1 (Lower bound for $\Phi^{\mathcal{X}}_{\mathrm{Proj}}(\delta)$-regret against non-convex losses)**.** *For any $T \ge 1$, $\delta \in (0, 1)$, and $L \ge 0$, there exists a sequence of L-Lipschitz and L-smooth non-convex loss functions $f^1, \dots, f^T$ on $\mathcal{X} = [-1, 1]$ such that for any algorithm that satisfies the linear span assumption, its $\Phi^{\mathcal{X}}_{\mathrm{Proj}}(\delta)$-regret on the loss sequence is $\mathrm{Reg}^T_{\mathrm{Proj},\delta} \ge \frac{\delta^2 LT}{2}$.*

5.2 Improved $\Phi^{\mathcal{X}}_{\mathrm{Proj}}(\delta)$-Regret in the Game Setting

Any online algorithm, as demonstrated by Theorem 4, suffers an $\Omega(\sqrt{T})$ $\Phi^{\mathcal{X}}_{\mathrm{Proj}}(\delta)$-regret even against linear loss functions. This lower bound, however, holds only in the *adversarial* setting where an adversary can choose arbitrary loss functions. In this section, we show an improved $O(T^{\frac{1}{4}})$ individual $\Phi^{\mathcal{X}}_{\mathrm{Proj}}(\delta)$-regret bound under a stronger smoothness assumption (Assumption 2) in the *game* setting, where players interact with each other using the same algorithm.

We study the Optimistic Gradient (OG) algorithm [Rakhlin and Sridharan, 2013], an optimistic variant of GD that has been shown to have improved individual *external* regret guarantee in the game setting [Syrgkanis et al., 2015]. The OG algorithm initializes $w^0 \in \mathcal{X}$ arbitrarily and $g^0 = 0$.

In each step $t \ge 1$, the algorithm plays x^t, receives feedback $g^t := \nabla f^t(x^t)$, and updates w^t, as follows:

$$x^t = \Pi_{\mathcal{X}}\left[w^{t-1} - \eta g^{t-1}\right], \quad w^t = \Pi_{\mathcal{X}}\left[w^{t-1} - \eta g^t\right]. \tag{OG}$$

We first prove an $O(\sqrt{T})$ $\Phi^{\mathcal{X}}_{\text{Proj}}(\delta)$-regret upper bound for OG in the adversarial setting.

Theorem 5 (Adversarial Regret Bound for OG)**.** *Let $\delta > 0$ and $T \in \mathbb{N}$. For convex and G-Lipschitz loss functions $\{f^t : \mathcal{X} \to \mathbb{R}\}_{t\in[T]}$, the $\Phi^{\mathcal{X}}_{\text{Proj}}(\delta)$-regret of (OG) with step size $\eta > 0$ is*

$$\text{Reg}^T_{\text{Proj},\delta} \le \frac{\delta D_{\mathcal{X}}}{\eta} + \eta \sum_{t=1}^{T} \left\|g^t - g^{t-1}\right\|^2.$$

Choosing step size $\eta = \frac{\sqrt{\delta D_{\mathcal{X}}}}{2G\sqrt{T}}$, we have $\text{Reg}^T_{\text{Proj},\delta} \le 4G\sqrt{\delta D_{\mathcal{X}} T}$.

In the analysis of Theorem 5 for the adversarial setting, the term $\|g^t - g^{t-1}\|^2$ can be as large as $4G^2$. In the game setting where every player i employs OG, g^t_i ,i.e., $-\nabla_{x_i} u_i(x)$, depends on other players' action x^t_{-i}. Note that the change of the players' actions $\|x^t - x^{t-1}\|^2$ is only $O(\eta^2)$. Such stability of the updates leads to an improved upper bound on $\|g^t_i - g^{t-1}_i\|^2$ and hence also an improved $O(T^{\frac{1}{4}})$ $\Phi^{\mathcal{X}}_{\text{Proj}}(\delta)$-regret for the player.

Theorem 6 (Improved Individual $\Phi^{\mathcal{X}}_{\text{Proj}}(\delta)$-Regret of OG in the Game Setting)**.** *In a G-Lipschitz L-smooth (in the sense of Assumption 2) differentiable game, when all players employ OG with step size $\eta > 0$, then for each player i, $\delta > 0$, and $T \ge 1$, their individual $\Phi^{\mathcal{X}_i}_{\text{Proj}}(\delta)$-regret denoted as $\text{Reg}^{T,i}_{\text{Proj},\delta}$ is*

$$\text{Reg}^{T,i}_{\text{Proj},\delta} \le \frac{\delta D}{\eta} + \eta G^2 + 3nL^2G^2\eta^3 T.$$

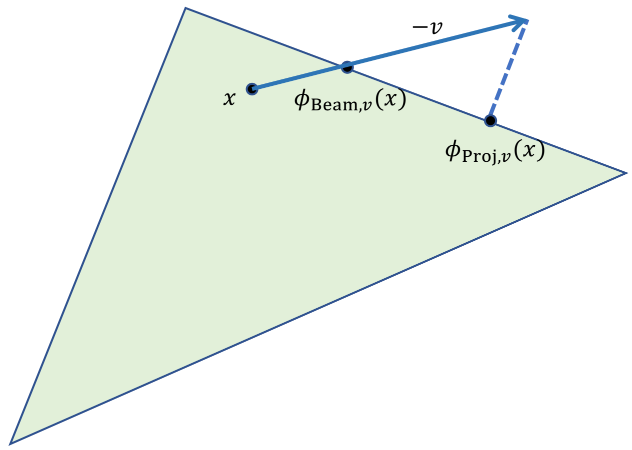
Choosing $\eta = \min\{(\delta D/(nL^2G^2T))^{\frac{1}{4}}, (\delta D)^{\frac{1}{2}}/G\}$, we have $\text{Reg}^{T,i}_{\text{Proj},\delta} \le 4(\delta D)^{\frac{3}{4}}(nL^2G^2T)^{\frac{1}{4}} + 2\sqrt{\delta D}G$. Furthermore, for any $\delta > 0$ and any $\varepsilon > \frac{\delta^2 L}{2}$, their empirical distribution of played strategy profiles converges to $(\varepsilon, \Phi_{\text{Proj}}(\delta))$-local equilibrium in $O(\max\{\frac{\delta D(nL^2G^2)^{\frac{1}{3}}}{(2\varepsilon-\delta^2 L)^{\frac{4}{3}}}, \frac{\sqrt{\delta D}G}{2\varepsilon-\delta^2 L}\}) = O(1/\varepsilon^{\frac{4}{3}})$ iterations.

6 Beam-Search Local Strategy Modifications and Local Equilibria

In the previous two sections, we have shown that GD achieves near-optimal performance for both $\Phi^{\mathcal{X}}_{\text{Int}}(\delta)$-regret and $\Phi^{\mathcal{X}}_{\text{Proj}}(\delta)$-regret. In this section, we introduce another natural set of local strategy modifications, $\Phi^{\mathcal{X}}_{\text{Beam}}(\delta)$, which is similar to $\Phi^{\mathcal{X}}_{\text{Proj}}(\delta)$. Specifically, the set $\Phi^{\mathcal{X}}_{\text{Beam}}(\delta)$ contains deviations that try to move as far as possible in a fixed direction (see Figure 3 for an illustration of the difference between $\phi_{\text{Beam},v}(x)$ and $\phi_{\text{Proj},v}(x)$):

$$\Phi^{\mathcal{X}}_{\text{Beam}}(\delta) := \{\phi_{\text{Beam},v}(x) = x - \lambda^* v : v \in B_d(\delta), \lambda^* = \max\{\lambda : x - \lambda v \in \mathcal{X}, \lambda \in [0,1]\}\}.$$

Figure 3: Illustration of $\phi_{\text{Proj},v}(x)$ and $\phi_{\text{Beam},v}(x)$



It is clear that $\|\phi_{\text{Beam},v}(x) - x\| \le \|v\| \le \delta$. We can similarly derive the notion of $\Phi^{\mathcal{X}}_{\text{Beam}}$-regret and $(\varepsilon, \Phi_{\text{Beam}}(\delta))$-local equilibrium. Surprisingly, we show that GD suffers linear $\Phi^{\mathcal{X}}_{\text{Beam}}(\delta)$-regret (proof deferred to Appendix D).

Theorem 7. *For any $\delta, \eta < \frac{1}{2}$ and $T \ge 1$, there exists a sequence of linear loss functions $\{f^t : \mathcal{X} \subseteq [0,1]^2 \to \mathbb{R}\}_{t \in [T]}$ such that GD with step size η suffers $\Omega(\delta T)$ $\Phi^{\mathcal{X}}_{\text{Beam}}(\delta)$-regret.*

7 Discussion and Future Directions

We have shown that (i) $\Phi^{\mathcal{X}}_{\text{Int}}(\delta)$-regret reduces to external regret; (ii) $\Phi^{\mathcal{X}}_{\text{Proj}}(\delta)$-regret is different from external regret and requires a new analysis of GD; (iii) GD fails to achieve no $\Phi^{\mathcal{X}}_{\text{Beam}}(\delta)$-regret. These results together show that, even for simple local strategy modification sets $\Phi(\delta)$, the landscape of efficient local $\Phi(\delta)$-regret minimization is already quite rich, and many basic and interesting questions remain open.

More local Φ-regret It will be interesting to investigate for what other local strategy modifications Φ, the Φ-regret can be minimized efficiently. One natural candidate is $\Phi^{\mathcal{X}}_{\text{Beam}}(\delta)$.

Improved $\Phi^{\mathcal{X}}_{\text{Proj}}(\delta)$-regret in games We show in Theorem 6 that the optimistic gradient (OG) dynamics give individual $\Phi^{\mathcal{X}}_{\text{Proj}}(\delta)$-regret of $O(T^{1/4})$. Could we design uncoupled learning dynamics with better individual regret guarantees, consequently leading to faster convergence to an $(\varepsilon, \Phi_{\text{Proj}}(\delta))$-local equilibrium?

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Contents

A Additional Preliminaries: Solution Concepts in Non-Concave Games

We present definitions of several solution concepts in the literature as well as the existence and computational complexity of each solution concept.

Definition 7 (Nash Equilibrium)**.** *In a differentiable game, a strategy profile* $x \in \prod_{j=1}^n \mathcal{X}_j$ *is a* Nash equilibrium (NE) *if and only if for every player* $i \in [n]$*,*

$$u_i(x_i', x_{-i}) \leq u_i(x), \forall x_i' \in \mathcal{X}_i$$

Definition 8 (Mixed Nash Equilibrium)**.** *In a differentiable game, a mixed strategy profile* $p \in \prod_{j=1}^n \Delta(\mathcal{X}_j)$ *(here we denote* $\Delta(\mathcal{X}_i)$ *as the set of probability measures over* $\mathcal{X}_i$*) is a* mixed Nash equilibrium (MNE) *if and only if for every player* $i \in [n]$*,*

$$u_i(p_i', p_{-i}) \leq u_i(p), \forall p_i' \in \Delta(\mathcal{X}_i)$$

Definition 9 ((Coarse) Correlated Equilibrium)**.** *In a differentiable game, a distribution* σ *over joint strategy profiles* $\Pi_{i=1}^n \mathcal{X}_i$ *is a* correlated equilibrium (CE) *if and only if for all player* $i \in [n]$*,*

$$\max_{\phi_i : \mathcal{X}_i \to \mathcal{X}_i} \mathbb{E}_{x \sim \sigma}[u_i(\phi_i(x_i), x_{-i})] \leq \mathbb{E}_{x \sim \sigma}[u_i(x)].$$

Similarly, a distribution σ *over joint strategy profiles* $\Pi_{i=1}^n \mathcal{X}_i$ *is a* coarse correlated equilibrium (CCE) *if and only if for all player* $i \in [n]$*,*

$$\max_{x_i' \in \mathcal{X}_i} \mathbb{E}_{x \sim \sigma}[u_i(x_i', x_{-i})] \leq \mathbb{E}_{x \sim \sigma}[u_i(x)].$$

Definition 10 (Strict Local Nash Equilibrium)**.** *In a differentiable game, a strategy profile* $x \in \prod_{j=1}^n \mathcal{X}_j$ *is a* strict local Nash equilibrium *if and only if for every player* $i \in [n]$*, there exists* $\delta > 0$ *such that*

$$u_i(x_i', x_{-i}) \leq u_i(x), \forall x_i' \in B_{d_i}(x_i, \delta) \cap \mathcal{X}_i.$$

Definition 11 (Second-order Local Nash Equilibrium)**.** *Consider a differentiable game where each utility function* $u_i(x_i, x_{-i})$ *is twice-differentiable with respect to* x_i *for any fixed* x_{-i}*. A strategy profile* $x \in \prod_{j=1}^n \mathcal{X}_j$ *is a* second-order local Nash equilibrium *if and only if for every player* $i \in [n]$*,* x_i *maximizes the second-order Taylor expansion of its utility functions at* x_i*, or formally,*

$$\langle \nabla_{x_i} u_i(x), x_i' - x_i \rangle + (x_i' - x_i)^\top \nabla_{x_i}^2 u_i(x)(x_i' - x_i) \leq 0, \forall x_i' \in \mathcal{X}_i.$$

Existence Mixed Nash equilibria exist in continuous games, and thus smooth differentiable games [Debreu, 1952, Glicksberg, 1952, Fan, 1953]. By definition, an MNE is also a CE and a CCE. This also proves existence of CE and CCE. In contrast, strict local Nash equilibria, second-order Nash equilibria, or (pure) Nash equilibria may not exist in a smooth non-concave game as we shown in the following example.

Example 3. *Consider a two-player zero-sum non-concave game: the action sets are* $\mathcal{X}_1 = \mathcal{X}_2 = [-1, 1]$ *and the utility functions are* $u_1(x_1, x_2) = -u_2(x_1, x_2) = (x_1 - x_2)^2$*. Let* $x = (x_1, x_2) \in \mathcal{X}_1 \times \mathcal{X}_2$ *be any strategy profile: if* $x_1 = x_2$*, then player 1 is not at a local maximum; if* $x_1 \neq x_2$*, then player 2 is not at a local maximum. Thus* x *is not a strict local Nash equilibrium. Since the utility function is quadratic, we conclude that the game also has no second-order local Nash equilibrium.*

Computational Complexity Consider a single-player smooth non-concave game with a quadratic utility function $f : \mathcal{X} \to \mathbb{R}$, then the problem of finding a *local* maximum of f can be reduced to the problem of computing a NE, a MNE, a CE, a CCE, a strict local Nash equilibrium, or a second-order local Nash equilibrium. Since computing a local maximum or checking if a given point is a local maximum is NP-hard [Murty and Kabadi, 1987], we know that the computational complexities of NE, MNE, CE, CCE, strict local Nash equilibria, and second-order local Nash equilibria are all NP-hard.

Representation Complexity Karlin [1959] present a two-player zero-sum non-concave game whose unique MNE has infinite support. Since in a two-player zero-sum game, the marginal distribution of a CE or a CCE is a MNE, it also implies that the representation complexity of any CE or CCE is infinite. We present the example in Karlin [1959] here for completeness and also prove that the game is Lipschitz and smooth.

Example 4 ([Karlin, 1959, Chapter 7.1, Example 3])**.** *We consider a two-player zero-sum game with action sets* $\mathcal{X}_1 = \mathcal{X}_2 = [0, 1]$*. Let* p *and* q *be two distributions over* $[0, 1]$*. The only requirement for* p *and* q *is that their cumulative distribution functions are not finite-step functions. For example, we can take* $p = q$ *to be the uniform distribution.*

Let μ_n *and* ν_n *denote the* n*-th moments of* p *and* q*, respectively. Define the utility function*

$$u(x, y) = u_1(x, y) = -u_2(x, y) = \sum_{n=0}^{\infty} \frac{1}{2^n}(x^n - \mu_n)(y^n - \nu_n), \quad 0 \le x, y \le 1.$$

Claim 1. *The game in Example 4 is* 2*-Lipschitz and* 6*-smooth, and* (p, q) *is its unique (mixed) Nash equilibrium.*

Proof. Fix any $y \in [0, 1]$, since $|\frac{1}{2^n}(y^n - \nu_n)nx^{n-1}| \le \frac{n}{2^n}$, the series of $\nabla_x u(x, y)$ is uniformly convergent. We have $|\nabla_x u(x, y)| \le \sum_{n=0}^{\infty} \frac{n}{2^n} \le 2, \quad y \in [0, 1]$. Similarly, we have $|\nabla_x^2 u(x, y)| \le \sum_{n=0}^{\infty} \frac{n^2}{2^n} \le 6$ for all $y \in [0, 1]$. By symmetry, we also have $|\nabla_y(x, y)| \le 2$ and $|\nabla_y^2(x, y)| \le 6$ for all $x, y \in [0, 1]$. Thus the game is 2-Lispchitz and 6-smooth.

Since $|\frac{1}{2^n}(x^n - \mu_n)(y^n - \nu_n)| \le \frac{1}{2^n}$, the series of $u(x, y)$ is absolutely and uniformly convergent. We have

$$\int_0^1 u(x, y) \mathrm{d}F_p(x) = \sum_{n=0}^{\infty} \frac{1}{2^n}(y^n - \nu_n) \int_0^1 (x^n - \mu_n) \mathrm{d}F_p(x) \equiv 0,$$

$$\int_0^1 u(x, y) F_q(y) = \sum_{n=0}^{\infty} \frac{1}{2^n}(x^n - \mu_n) \int_0^1 (y^n - \nu_n) \mathrm{d}F_q(y) \equiv 0.$$

In particular, (p, q) is a mixed Nash equilibrium and the value of the game is 0. Suppose (p', q') is also a mixed Nash equilibrium. Then (p, q') is a mixed Nash equilibrium. Note that p supports on every point in $[0, 1]$. As a consequence, we have

$$0 \equiv \int_0^1 u(x, y)\mathrm{d}F_{q'}(y) = \sum_{n=0}^{\infty} \frac{1}{2^n}(x^n - \mu_n)(\nu'_n - \nu_n),$$

where ν'_n is the n-th moment of q'. Since the series vanished identically, the coefficients of each power of x must vanish. Thus $\nu'_n = \nu_n$ and $q' = q$. Similarly, we have $p' = p$ and the mixed Nash equilibrium is unique. □

B Proof of Lemma 1

Let $\{f^t\}_{t\in[T]}$ be a sequence of non-convex L-smooth loss functions satisfying Assumption 1. Let $\{x^t\}_{t\in[T]}$ be the iterates produced by $\mathcal{A}$ against $\{f^t\}_{t\in[T]}$. Then $\{x^t\}_{t\in[T]}$ is also the iterates produced by $\mathcal{A}$ against a sequence of linear loss functions $\{\langle \nabla f^t(x^t), \cdot \rangle\}$. For the latter, we know

$$\max_{\phi\in\Phi^{\mathcal{X}}(\delta)} \sum_{t=1}^{T} \left\langle \nabla f^t(x^t), x^t - \phi(x^t) \right\rangle \le \mathrm{Reg}^T_{\Phi^{\mathcal{X}}(\delta)}.$$

Then using L-smoothness of $\{f^t\}$ and the fact that $\|\phi(x) - x\| \le \delta$ for all $\phi \in \Phi(\delta)$, we get

$$\begin{aligned}\max_{\phi\in\Phi^{\mathcal{X}}(\delta)} \sum_{t=1}^{T} f^t(x^t) - f^t(\phi(x^t)) &\le \max_{\phi\in\Phi^{\mathcal{X}}(\delta)} \sum_{t=1}^{T} \left(\left\langle \nabla f^t(x^t), x^t - \phi(x^t) \right\rangle + \frac{L}{2}\left\|x^t - \phi(x^t)\right\|^2 \right) \\ &\le \mathrm{Reg}^T_{\Phi^{\mathcal{X}}(\delta)} + \frac{\delta^2 LT}{2}.\end{aligned}$$

This completes the proof of the first part.

Let each player $i \in [n]$ employ algorithm $\mathcal{A}$ in a smooth game independently and produces iterates $\{x^t\}$. The averaged joint strategy profile σ^T that chooses x^t uniformly at random from $t \in [T]$ satisfies for any player $i \in [n]$,

$$\begin{aligned}&\max_{\phi\in\Phi^{\mathcal{X}_i}(\delta)} \mathbb{E}_{x\sim\sigma}[u_i(\phi(x_i), x_{-i})] - \mathbb{E}_{x\sim\sigma}[u_i(x)] \\ &= \max_{\phi\in\Phi^{\mathcal{X}_i}(\delta)} \frac{1}{T}\sum_{t=1}^{T} \left(u_i(\phi(x_i^t), x_{-i}^t) - u_i(x^t)\right) \\ &\le \frac{\mathrm{Reg}^T_{\Phi^{\mathcal{X}_i}(\delta)}}{T} + \frac{\delta^2 L}{2}.\end{aligned}$$

Thus σ^T is a $(\max_{i\in[n]}\{\mathrm{Reg}^T_{\Phi^{\mathcal{X}_i}(\delta)}\} \cdot T^{-1} + \frac{\delta^2 L}{2}, \Phi(\delta))$-local equilibrium. This completes the proof of the second part.

C Missing Proofs in Section 5

C.1 Proof of Theorem 4

Our proof technique comes from the standard one used in multi-armed bandits [Auer et al., 2002, Theorem 5.1]. Suppose that $f^t(x) = g^t x$. We construct two possible environments. In the first environment, $g^t = G$ with probability $\frac{1+\epsilon}{2}$ and $g^t = -G$ with probability $\frac{1-\epsilon}{2}$; in the second environment, $g^t = G$ with probability $\frac{1-\epsilon}{2}$ and $g^t = -G$ with probability $\frac{1+\epsilon}{2}$. We use $\mathbb{E}_i$ and $\mathbb{P}_i$ to denote the expectation and probability measure under environment i, respectively, for $i = 1, 2$. Suppose that the true environment is uniformly chosen from one of these two environments. Below, we show that the expected regret of the learner is at least $\Omega(\delta G\sqrt{T})$.

Define $N_+ = \sum_{t=1}^T \mathbb{I}\{x^t \geq 0\}$ be the number of times x^t is non-negative, and define $f^{1:T} = (f^1, \dots, f^T)$. Then we have

$$\begin{aligned}
|\mathbb{E}_1[N_+] - \mathbb{E}_2[N_+]| &= \left| \sum_{f^{1:T}} \left(\mathbb{P}_1(f^{1:T})\mathbb{E}\left[N_+ \mid f^{1:T}\right] - \mathbb{P}_2(f^{1:T})\mathbb{E}\left[N_+ \mid f^{1:T}\right] \right) \right| \\
&\hspace{15em} \text{(enumerate all possible sequences of } f^{1:T}\text{)} \\
&\leq T \sum_{f^{1:T}} \left|\mathbb{P}_1(f^{1:T}) - \mathbb{P}_2(f^{1:T})\right| \\
&= T\|\mathbb{P}_1 - \mathbb{P}_2\|_{\mathrm{TV}} \\
&\leq T\sqrt{(2\ln 2)\mathrm{KL}(\mathbb{P}_1, \mathbb{P}_2)} \qquad \text{(Pinsker's inequality)} \\
&= T\sqrt{(2\ln 2)T \cdot \mathrm{KL}\left(\text{Bernoulli}\left(\frac{1+\epsilon}{2}\right), \text{Bernoulli}\left(\frac{1-\epsilon}{2}\right)\right)} \\
&= T\sqrt{(2\ln 2)T\epsilon \ln \frac{1+\epsilon}{1-\epsilon}} \leq T\sqrt{(4\ln 2)T\epsilon^2}. \qquad (1)
\end{aligned}$$

In the first environment, we consider the regret with respect to $v = \delta$. Then we have

$$\begin{aligned}
\mathbb{E}_1\left[\mathrm{Reg}^T_{\mathrm{Proj},\delta}\right] &\geq \mathbb{E}_1\left[\sum_{t=1}^T f^t(x^t) - f^t(\Pi_{\mathcal{X}}[x^t - \delta])\right] = \mathbb{E}_1\left[\sum_{t=1}^T g^t(x^t - \Pi_{\mathcal{X}}[x^t - \delta])\right] \\
&= \mathbb{E}_1\left[\sum_{t=1}^T \epsilon G(x^t - \Pi_{\mathcal{X}}[x^t - \delta])\right] \geq \epsilon\delta G\mathbb{E}_1\left[\sum_{t=1}^T \mathbb{I}\{x^t \geq 0\}\right] = \epsilon\delta G\mathbb{E}_1\left[N_+\right],
\end{aligned}$$

where in the last inequality we use the fact that if $x^t \geq 0$ then $x^t - \Pi_{\mathcal{X}}[x^t - \delta] = x^t - (x^t - \delta) = \delta$ because $D \geq \delta$. In the second environment, we consider the regret with respect to $v = -\delta$. Then similarly, we have

$$\mathbb{E}_2\left[\mathrm{Reg}^T_{\mathrm{Proj},\delta}\right] \ge \mathbb{E}_2\left[\sum_{t=1}^T f^t(x^t) - f^t(\Pi_{\mathcal{X}}[x^t+\delta])\right] = \mathbb{E}_2\left[\sum_{t=1}^T g^t(x^t - \Pi_{\mathcal{X}}[x^t+\delta])\right]$$
$$= \mathbb{E}_2\left[\sum_{t=1}^T -\epsilon G(x^t - \Pi_{\mathcal{X}}[x^t+\delta])\right] \ge \epsilon\delta G \mathbb{E}_2\left[\sum_{t=1}^T \mathbb{I}\{x^t < 0\}\right] = \epsilon\delta G\left(T - \mathbb{E}_2\left[N_+\right]\right).$$

Summing up the two inequalities, we get

$$\frac{1}{2}\left(\mathbb{E}_1\left[\mathrm{Reg}^T_{\mathrm{Proj},\delta}\right] + \mathbb{E}_2\left[\mathrm{Reg}^T_{\mathrm{Proj},\delta}\right]\right) \ge \frac{1}{2}\left(\epsilon\delta GT + \epsilon\delta G(\mathbb{E}_1[N_+] - \mathbb{E}_2[N_+])\right)$$
$$\ge \frac{1}{2}\left(\epsilon\delta GT - \epsilon\delta GT\epsilon\sqrt{(4\ln 2)T}\right). \qquad \text{(by (1))}$$

Choosing $\epsilon = \frac{1}{\sqrt{(16\ln 2)T}}$, we can lower bound the last expression by $\Omega(\delta G\sqrt{T})$. The theorem is proven by noticing that $\frac{1}{2}\left(\mathbb{E}_1\left[\mathrm{Reg}^T_{\mathrm{Proj},\delta}\right] + \mathbb{E}_2\left[\mathrm{Reg}^T_{\mathrm{Proj},\delta}\right]\right)$ is the expected regret of the learner.

C.2 Proof of Proposition 1

Proof. Consider $f : [-1,1] \to \mathbb{R}$ such that $f(x) = -\frac{L}{2}x^2$ and let $f^t = f$ for all $t \in [T]$. Then any first-order methods that satisfy the linear span assumption with initial point $x^1 = 0$ will produce $x^t = 0$ for all $t \in [T]$. The $\Phi^{\mathcal{X}}_{\mathrm{Proj}}(\delta)$-regret is thus $\sum_{t=1}^T (f(0) - f(\delta)) = \frac{\delta^2 LT}{2}$. □

C.3 Proof of Theorem 3

Proof. Fix any deviation v that is bounded by δ. Let us define $p^0 = w^0$ and $p^t = \Pi_{\mathcal{X}}[x^t - v]$. Following standard analysis of OG [Rakhlin and Sridharan, 2013], we have

$$\sum_{t=1}^T f^t(x^t) - f^t(p^t) \le \sum_{t=1}^T \left\langle \nabla f^t(x^t), x^t - p^t \right\rangle$$
$$\le \sum_{t=1}^T \frac{1}{2\eta}\left(\left\|w^{t-1} - p^t\right\|^2 - \left\|w^t - p^t\right\|^2\right) + \eta\left\|g^t - g^{t-1}\right\|^2 - \frac{1}{2\eta}\left(\left\|x^t - w^t\right\|^2 + \left\|x^t - w^{t-1}\right\|^2\right)$$
$$\le \sum_{t=1}^T \left(\frac{1}{2\eta}\left\|w^{t-1} - p^t\right\|^2 - \frac{1}{2\eta}\left\|w^{t-1} - p^{t-1}\right\|^2 + \eta\left\|g^t - g^{t-1}\right\|^2 - \frac{1}{2\eta}\left\|x^t - w^{t-1}\right\|^2\right) \qquad (2)$$

Now we apply similar analysis from Theorem 3 to upper bound the term $\|w^{t-1}-p^t\|^2 - \|w^{t-1}-p^{t-1}\|^2$:

$$\begin{aligned}
&\left\|w^{t-1}-p^t\right\|^2 - \left\|w^{t-1}-p^{t-1}\right\|^2 \\
&= \left\langle p^{t-1}-p^t, 2w^{t-1}-p^{t-1}-p^t \right\rangle \\
&= \left\langle p^{t-1}-p^t, 2w^{t-1}-2p^t \right\rangle - \left\|p^t-p^{t-1}\right\|^2 \\
&= 2\left\langle p^{t-1}-p^t, v \right\rangle + 2\left\langle p^{t-1}-p^t, w^{t-1}-v-p^t \right\rangle - \left\|p^t-p^{t-1}\right\|^2 \\
&= 2\left\langle p^{t-1}-p^t, v \right\rangle + 2\left\langle p^{t-1}-p^t, x^t-v-p^t \right\rangle + 2\left\langle p^{t-1}-p^t, w^{t-1}-x^t \right\rangle - \left\|p^t-p^{t-1}\right\|^2 \\
&\le 2\left\langle p^{t-1}-p^t, v \right\rangle + \left\|x^t-w^{t-1}\right\|^2,
\end{aligned}$$

where in the last-inequality we use $\langle p^{t-1}-p^t, x^t-u-p^t\rangle \le 0$ since $p^t = \Pi_{\mathcal{X}}[x^t-v]$ and $\mathcal{X}$ is a compact convex set; we also use $2\langle a,b\rangle - b^2 \le a^2$. In the analysis above, unlike the analysis of GD where we drop the negative term $-\|p^t-p^{t-1}\|^2$, we use $-\|p^t-p^{t-1}\|^2$ to get a term $\|x^t-w^{t-1}\|^2$ which can be canceled by the last term in (2).

Now we combine the above two inequalities. Since the term $\|x^t-w^{t-1}\|^2$ cancels out and $2\langle p^{t-1}-p^t, v\rangle$ telescopes, we get

$$\sum_{t=1}^{T} f^t(x^t) - f^t(p^t) \le \frac{\langle p^0-p^T, u\rangle}{\eta} + \sum_{t=1}^{T} \eta\left\|g^t-g^{t-1}\right\|^2 \le \frac{\delta D_{\mathcal{X}}}{\eta} + \eta\sum_{t=1}^{T}\left\|g^t-g^{t-1}\right\|^2.$$

□

C.4 Proof of Theorem 6

Proof. Let us fix any player $i \in [n]$ in the smooth game. In every step t, player i's loss function $f^t : \mathcal{X}_i \to \mathbb{R}$ is $\langle -\nabla_{x_i} u_i(x^t), \cdot\rangle$ determined by their utility function u_i and all players' actions x^t. Therefore, their gradient feedback is $g^t = -\nabla_{x_i} u_i(x^t)$. For all $t \ge 2$, we have

$$\begin{aligned}
\left\|g^t-g^{t-1}\right\|^2 &= \left\|\nabla u_i(x^t) - \nabla u_i(x^{t-1})\right\|^2 \\
&\le L^2\left\|x^t-x^{t-1}\right\|^2 \\
&= L^2\sum_{i=1}^{n}\left\|x_i^t-x_i^{t-1}\right\|^2 \\
&\le 3L^2\sum_{i=1}^{n}\left(\left\|x_i^t-w_i^t\right\|^2 + \left\|w_i^t-w_i^{t-1}\right\|^2 + \left\|w_i^{t-1}-x_i^{t-1}\right\|^2\right) \\
&\le 3nL^2\eta^2G^2,
\end{aligned}$$

where we use L-smoothness of the utility function u_i in the first inequality; we use the update rule of OG and the fact that gradients are bounded by G in the last inequality.

Applying the above inequality to the regret bound obtained in Theorem 5, the individual $\Phi^{\mathcal{X}}_{\mathrm{Proj}}(\delta)$-regret of player i is upper bounded by

$$\mathrm{Reg}^{T,i}_{\mathrm{Proj},\delta} \le \frac{\delta D}{\eta} + \eta G^2 + 3nL^2G^2\eta^3 T.$$

Choosing $\eta = \min\{(\delta D/(nL^2G^2T))^{\frac{1}{4}}, (\delta D)^{\frac{1}{2}}/G\}$, we have $\mathrm{Reg}^{T,i}_{\mathrm{Proj},\delta} \le 4(\delta D)^{\frac{3}{4}}(nL^2G^2T)^{\frac{1}{4}} + 2\sqrt{\delta D}G$. Using Lemma 1, we have the empirical distribution of played strategy profiles converges to $(\varepsilon, \Phi_{\mathrm{Proj}}(\delta))$-local equilibrium in $O(\max\{\frac{\delta D(nL^2G^2)^{\frac{1}{3}}}{(2\varepsilon-\delta^2L)^{\frac{4}{3}}}, \frac{\sqrt{\delta D}G}{2\varepsilon-\delta^2 L}\})$ iterations. □

D Proof of Theorem 7

Let $\mathcal{X} \subset \mathbb{R}^2$ be a triangle region with vertices $A = (0,0)$, $B = (1,1)$, $C = (\delta, 0)$. Consider $v = (-\delta, 0)$. The initial point is $x_1 = (0,0)$.

The adversary will choose ℓ_t adaptively so that x_t remains on the boundary of $\mathcal{X}$ and cycles clockwise (i.e., $A \to \cdots \to B \to \cdots \to C \to \cdots \to A \to \cdots$). To achieve this, the adversary will repeat the following three phases:

1. Keep choosing $\ell_t = u_{\overrightarrow{BA}}$ ($u_{\overrightarrow{BA}}$ denotes the unit vector in the direction of $\overrightarrow{BA}$) until x_{t+1} reaches B.
2. Keep choosing $\ell_t = u_{\overrightarrow{CB}}$ until x_{t+1} reaches C.
3. Keep choosing $\ell_t = u_{\overrightarrow{AC}}$ until x_{t+1} reaches A.

In Phase 1, $x_t \in \overline{AB}$. By the choice of $v = (-\delta, 0)$, we have $x_t - \phi_v(x_t) = (-\delta(1 - x_{t,1}), 0)$, and the instantaneous regret is $\frac{\delta(1-x_{t,1})}{\sqrt{2}} \ge 0$.

In Phase 2, $x_t \in \overline{BC}$. By the choice of $v = (-\delta, 0)$, we have $x_t - \phi_v(x_t) = (0,0)$, and the instantaneous regret is 0.

In Phase 3, $x_t \in \overline{CA}$. By the choice of $v = (-\delta, 0)$, we have $x_t - \phi_v(x_t) = (-\delta + x_{t,1}, 0)$, and the instantaneous regret is $-\delta + x_{t,1} \le 0$.

In each cycle, the number of rounds in Phase 1 is of order $\Theta(\frac{\sqrt{2}}{\eta})$, the number of rounds in Phase 2 is between $O(\frac{1}{\eta})$ and $O(\frac{\sqrt{2}}{\eta})$, the number of rounds in Phase 3 is of order $\Theta(\frac{\delta}{\eta})$.

Therefore, the cumulative regret in each cycle is roughly

$$\frac{\sqrt{2}}{\eta} \times \frac{0.5\delta}{\sqrt{2}} + 0 + \frac{\delta}{\eta}(-0.5\delta) = \frac{0.5\delta - 0.5\delta^2}{\eta}.$$

On the other hand, the number of cycles is no less than $\frac{T}{\frac{\sqrt{2}}{\eta}+\frac{\sqrt{2}}{\eta}+\frac{\delta}{\eta}} = \Theta(\eta T)$. Overall, the cumulative regret is at least $\frac{0.5\delta-0.5\delta^2}{\eta} \times \Theta(\eta T) = \Theta(\delta T)$ as long as $\delta < 0.5$.

E Hardness in the Global Regime

In the local regime $\delta \le \sqrt{2\varepsilon/L}$, (ε, δ)-local Nash equilibrium is intractable and we have shown polynomial-time algorithms for computing the weaker notions of $(\varepsilon, \Phi_{\mathrm{Int}}(\delta))$ and $(\varepsilon, \Phi_{\mathrm{Proj}}(\delta))$-local equilibrium. A natural question is whether correlation enables efficient computation of $(\varepsilon, \Phi(\delta))$-local equilibrium when δ is in the global regime, i.e., $\delta = \Omega(\sqrt{d})$. In this section, we prove both computational hardness and a query complexity lower bound for both $(\varepsilon, \Phi_{\text{Int}}(\delta))$ and $(\varepsilon, \Phi_{\text{Proj}}(\delta))$-local equilibrium in the global regime

To prove the lower bound results, we only require a single-player game. The problem of computing a $(\varepsilon, \Phi(\delta))$-local equilibrium becomes: given scalars $\varepsilon, \delta, G, L > 0$ and a polynomial-time Turing machine $\mathcal{C}_f$ evaluating a G-Lipschitz and L-smooth function $f : [0,1]^d \to [0,1]$ and its gradient $\nabla f : [0,1]^d \to \mathbb{R}^d$, we are asked to output a distribution σ that is a $(\varepsilon, \Phi(\delta))$-local equilibrium or $\perp$ if such equilibrium does not exist.

Hardness of $(\varepsilon, \Phi_{\text{Int}}(\delta))$-local equilibrium in the global regime When $\delta = \sqrt{d}$, which equals to the diameter D of $[0,1]^d$, then the problem of finding an $(\varepsilon, \Phi_{\text{Int}}(\delta))$-local equilibrium is equivalent to finding a (ε, δ)-local minimum of f: assume σ is an $(\varepsilon, \Phi_{\text{Int}}(\delta))$-local equilibrium of f, then there exists $x \in [0,1]^d$ in the support of σ such that

$$f(x) - \min_{x^* \in [0,1]^d} f(x^*) \leq \varepsilon.$$

Then hardness of finding an $(\varepsilon, \Phi_{\text{Int}}(\delta))$-local equilibrium follows from hardness of finding a (ε, δ)-local minimum of f [Daskalakis et al., 2021b]. The following Theorem is a corollary of Theorem 10.3 and 10.4 in [Daskalakis et al., 2021b].

Theorem 8 (Hardness of $(\varepsilon, \Phi_{\text{Int}}(\delta))$-local equilibrium in the global regime)**.** *In the worst case, the following two holds.*

- *Computing an $(\varepsilon, \Phi_{\text{Int}}(\delta))$-local equilibrium for a game on $\mathcal{X} = [0,1]^d$ with $G = \sqrt{d}$, $L = d$, $\varepsilon \leq \frac{1}{24}$, $\delta = \sqrt{d}$ is NP-hard.*
- *$\Omega(2^d/d)$ value/gradient queries are needed to determine an $(\varepsilon, \Phi_{\text{Int}}(\delta))$-local equilibrium for a game on $\mathcal{X} = [0,1]^d$ with $G = \Theta(d^{15})$, $L = \Theta(d^{22})$, $\varepsilon < 1$, $\delta = \sqrt{d}$.*

Hardness of $(\varepsilon, \Phi^{\mathcal{X}}_{\text{Proj}}(\delta))$-local equilibrium in the global regime The proofs of Theorem 9 and Corollary 1 can be found in the next section.

Theorem 9 (Hardness of $(\varepsilon, \Phi_{\text{Proj}}(\delta))$-local equilibrium in the global regime)**.** *In the worst case, the following two holds.*

- *Computing an $(\varepsilon, \Phi_{\text{Proj}}(\delta))$-local equilibrium for a game on $\mathcal{X} = [0,1]^d$ with $G = \Theta(d^{15})$, $L = \Theta(d^{22})$, $\varepsilon < 1$, $\delta = \sqrt{d}$ is NP-hard.*
- *$\Omega(2^d/d)$ value/gradient queries are needed to determine an $(\varepsilon, \Phi_{\text{Proj}}(\delta))$-local equilibrium for a game on $\mathcal{X} = [0,1]^d$ with $G = \Theta(d^{15})$, $L = \Theta(d^{22})$, $\varepsilon < 1$, $\delta = \sqrt{d}$.*

The hardness of computing $(\varepsilon, \Phi(\delta))$-local equilibrium also implies lower bound on $\Phi^{\mathcal{X}}_{\text{Proj}}(\delta)$-regret in the global regime.

Corollary 1 (Lower bound of $\Phi^{\mathcal{X}}_{\text{Proj}}(\delta)$-regret against non-convex functions)**.** *In the worst case, the $\Phi^{\mathcal{X}}_{\text{Proj}}(\delta)$-regret of any online algorithm is at least $\Omega(2^d/d, T)$ even for loss functions $f : [0,1]^d \to [0,1]$ with $G, L = \text{poly}(d)$ and $\delta = \sqrt{d}$.*

E.1 Proof of Theorem 9

We will reduce the problem of finding $(\varepsilon, \Phi_{\text{Proj}}(\delta))$-local equilibrium in smooth games to finding a satisfying assignment of a boolean function, which is NP-complete.

Fact 1. *Given only* black-box access *to a boolean formula $\phi : \{0,1\}^d \to \{0,1\}$, at least $\Omega(2^d)$ queries are needed in order to determine whether ϕ admits a satisfying assignment x^* such that $\phi(x^*) = 1$. The term* black-box access *refers to the fact that the clauses of the formula are not given and the only way to determine whether a specific boolean assignment is a satisfying is by querying the specific binary string. Moreover, the problem of finding a satisfying assigment of a general boolean function is NP-hard.*

We revisit the construction of the hard instance in the proof of [Daskalakis et al., 2021b, Theorem 10.4] and use its specific structures. Given black-box access to a boolean formula ϕ as described in Fact 1, following [Daskalakis et al., 2021b], we construct the function $f_\phi(x) : [0,1]^d \to [0,1]$ as follows:

1. for each corner $v \in V = \{0,1\}^d$ of the $[0,1]^d$ hypercube, we set $f_\phi(x) = 1 - \phi(x)$.
2. for the rest of the points $x \in [0,1]^d / V$, we set $f_\phi(x) = \sum_{v \in V} P_v(x) \cdot f_\phi(v)$ where $P_v(x)$ are non-negative coefficients defined in [Daskalakis et al., 2021b, Definition 8.9].

The function f_ϕ satisfies the following properties:

1. if ϕ is not satisfiable, then $f_\phi(x) = 1$ for all $x \in [0,1]^d$ since $f_\phi(v) = 1$ for all $v \in V$; if ϕ has a satisfying assignment v^*, then $f_\phi(v^*) = 0$.
2. f_ϕ is $\Theta(d^{12})$-Lipschitz and $\Theta(d^{25})$-smooth.
3. for any point $x \in [0,1]^d$, the set $V(x) := \{v \in V : P_v(x) \neq 0\}$ has cardinality at most $d+1$ while $\sum_{v \in V} P_v(x) = 1$; any value / gradient query of f_ϕ can be simulated by $d+1$ queries on ϕ.

In the case there exists a satisfying argument v^*, then $f_\phi(v^*) = 0$. Define the deviation e so that $e[i] = 1$ if $v^*[i] = 0$ and $e[i] = -1$ if $v^*[i] = 1$. It is clear that $\|e\| = \sqrt{d} = \delta$. By properties of projection on $[0,1]^d$, for any $x \in [0,1]^d$, we have $\Pi_{[0,1]^n}[x - v] = v^*$. Then any $(\varepsilon, \Phi_{\text{Proj}}(\delta))$-local equilibrium σ must include some $x^* \in \mathcal{X}$ with $f_\phi(x^*) < 1$ in the support, since $\varepsilon < 1$. In case there exists an algorithm $\mathcal{A}$ that computes an $(\varepsilon, \Phi_{\text{Proj}}(\delta))$-local equilibrium, $\mathcal{A}$ must have queried some x^* with $f_\phi(x^*) < 1$. Since $f_\phi(x^*) = \sum_{v \in V(x^*)} P_v(x^*) f_\phi(v) < 1$, there exists $\hat{v} \in V(x^*)$ such that $f_\phi(\hat{v}) = 0$. Since $|V(x^*)| \leq d+1$, it takes addition $d+1$ queries to find $\hat{v}$ with $f_\phi(\hat{v}) = 0$. By Fact 1 and the fact that we can simulate every value / gradient query of f_ϕ by $d+1$ queries on ϕ, $\mathcal{A}$ makes at least $\Omega(2^d/d)$ value / gradient queries.

Suppose there exists an algorithm $\mathcal{B}$ that outputs an $(\varepsilon, \Phi_{\text{Proj}}(\delta))$-local equilibrium σ in time $T(\mathcal{B})$ for $\varepsilon < 1$ and $\delta = \sqrt{d}$. We construct another algorithm $\mathcal{C}$ for SAT that terminates in time $T(\mathcal{B}) \cdot \text{poly}(d)$. $\mathcal{C}$: (1) given a boolean formula ϕ, construct f_ϕ as described above; (2) run $\mathcal{B}$ and get output σ (3) check the support of σ to find $v \in \{0,1\}^d$ such that $f_\phi(v) = 0$; (3) if finds

$v \in \{0,1\}^d$ such that $f_\phi(v) = 0$, then ϕ is satisfiable, otherwise ϕ is not satisfiable. Since we can evaluate f_ϕ and ∇f_ϕ in $\mathrm{poly}(d)$ time and the support of σ is smaller than $T(\mathcal{B})$, the algorithm $\mathcal{C}$ terminates in time $O(T(\mathcal{B}) \cdot \mathrm{poly}(d))$. The above gives a polynomial time reduction from SAT to $(\varepsilon, \Phi_{\mathrm{Proj}}(\delta))$-local equilibrium and proves the NP-hardness of the latter problem.

E.2 Proof of Corollary 1

Let $\phi : \{0,1\}^d \to \{0,1\}$ be a boolean formula and define $f_\phi : [0,1]^d \to [0,1]$ the same as that in Theorem 9. We know f_ϕ is $\Theta(\mathrm{poly}(d))$-Lipschitz and $\Theta(\mathrm{poly}(d))$-smooth. Now we let the adversary picks f_ϕ in each time. For any $T \le O(2^d/d)$, in case there exists an online learning algorithm with $\mathrm{Reg}^T_{\mathrm{Proj},\delta} < \frac{T}{2}$, then $\sigma := \frac{1}{T}\sum_{t=1}^{T} 1_{x^t}$ is an $(\frac{1}{2}, \delta)$-local equilibrium. Applying Theorem 9 and the fact that in this case $\mathrm{Reg}^T_{\mathrm{Proj},\delta}$ is non-decreasing with respect to T concludes the proof.

F Removing the D dependence

For the regime $\delta \le D_{\mathcal{X}}$ which we are more interested in, the lower bound in Theorem 4 is $\Omega(G\delta\sqrt{T})$ while the upper bound in Theorem 3 is $O(G\sqrt{\delta D_{\mathcal{X}} T})$. They are not tight especially when $D_{\mathcal{X}} \gg \delta$. A natural question is: *which of them is the tight bound?* We conjecture that the lower bound is tight. In fact, for the special case where the feasible set $\mathcal{X}$ is a *box*, we have a way to obtain a $D_{\mathcal{X}}$-independent bound $O(d^{\frac{1}{4}} G\delta\sqrt{T})$, which is tight when $d = 1$. Below, we first describe the improved strategy in 1-dimension. Then we show how to extend it to the d-dimensional box setting.

F.1 One-Dimensional Case

In one-dimension, we assume that $\mathcal{X} = [a, b]$ for some $b - a \ge 2\delta$ (if $b - a \le 2\delta$, then our original bound in Theorem 3 is already of order $G\delta\sqrt{T}$). We first investigate the case where $f^t(x)$ is a linear function, i.e., $f^t(x) = g^t x$ for some $g^t \in [-G, G]$. The key idea is that we will only select x^t from the two intervals $[a, a+\delta]$ and $[b-\delta, b]$, and never play $x^t \in (a+\delta, b-\delta)$. To achieve so, we concatenate these two intervals, and run an algorithm in this region whose diameter is only 2δ. The key property we would like to show is that the regret is preserved in this modified problem.

More precisely, given the original feasible set $\mathcal{X} = [a, b]$, we create a new feasible set $\mathcal{Y} = [-\delta, \delta]$ and apply our algorithm GD in this new feasible set. The loss function is kept as $f^t(x) = g^t x$. Whenever the algorithm for $\mathcal{Y}$ outputs $y^t \in [-\delta, 0]$, we play $x^t = y^t + a + \delta$ in $\mathcal{X}$; whenever it outputs $y^t \in (0, \delta]$, we play $x^t = y^t + b - \delta$. Below we show that the regret is the same in these two problems. Notice that when $y^t \le 0$, we have for any $v \in [-\delta, \delta]$,

$$
\begin{aligned}
x^t - \Pi_{\mathcal{X}}[x^t - v] &= x^t - \max\left(\min\left(x^t - v, b\right), a\right) \\
&= x^t - \max\left(x^t - v, a\right) \;\; (x^t - v = y^t + a + \delta - v \le a + 2\delta \le b \text{ always holds}) \\
&= y^t + a + \delta - \max\left(y^t + a + \delta - v, a\right) \\
&= y^t - \max\left(y^t - v, -\delta\right) \\
&= y^t - \max\left(\min\left(y^t - v, \delta\right), -\delta\right) \qquad (y^t - v \le \delta \text{ always holds}) \\
&= y^t - \Pi_{\mathcal{Y}}[y^t - v]
\end{aligned}
$$

Similarly, when $y^t > 0$, we can follow the same calculation and prove $x^t - \Pi_{\mathcal{X}}[x^t - v] = y^t - \Pi_{\mathcal{Y}}[y^t - v]$. Thus, the regret in the two problems:

$$
g^t\left(x^t - \Pi_{\mathcal{X}}[x^t - v]\right) \quad \text{and} \quad g^t\left(y^t - \Pi_{\mathcal{Y}}[y^t - v]\right)
$$

are exactly the same for any v. Finally, observe that the diameter of $\mathcal{Y}$ is only of order $O(\delta)$. Thus, the upper bound in Theorem 3 would give us an upper bound of $O(G\sqrt{\delta \cdot \delta T}) = O(G\delta\sqrt{T})$.

For convex f^t, we run the algorithm above with $g^t = \nabla f^t(x^t)$. Then by convexity we have

$$
f^t(x^t) - f^t(\Pi_{\mathcal{X}}[x^t - v]) \le g^t(x^t - \Pi_{\mathcal{X}}[x^t - v]) = g^t(y^t - \Pi_{\mathcal{Y}}[y^t - v]),
$$

so the regret in the modified problem (which is $O(G\delta\sqrt{T})$) still serves as a regret upper bound for the original problem.

F.2 d-Dimensional Box Case

A d-dimensional box is of the form $\mathcal{X} = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_d, b_d]$. The box case is easy to deal with because we can decompose the regret into individual components in each dimension. Namely, we have

$$
\begin{aligned}
f^t(x^t) - f^t(\Pi_{\mathcal{X}}[x^t - v]) &\le \nabla f^t(x^t)^\top \left(x^t - \Pi_{\mathcal{X}}[x^t - v]\right) \\
&= \sum_{i=1}^d g_i^t \left(x_i^t - \Pi_{\mathcal{X}_i}[x_i^t - v_i]\right)
\end{aligned}
$$

where we define $\mathcal{X}_i = [a_i, b_i]$, $g^t = \nabla f^t(x^t)$, and use subscript i to indicate the i-th component of a vector. The last equality above is guaranteed by the box structure. This decomposition allows as to view the problem as d independent 1-dimensional problems.

Now we follow the strategy described in Section F.1 to deal with individual dimensions (if $b_i - a_i < 2\delta$ then we do not modify $\mathcal{X}_i$; otherwise, we shrink $\mathcal{X}_i$ to be of length 2δ). Applying the analysis of Theorem 3 to each dimension, we get

$$\begin{aligned}
&\sum_{i=1}^{d} g_i^t \left(x_i^t - \Pi_{\mathcal{X}_i}[x_i^t - v_i]\right) \\
&\leq \sum_{i=1}^{d} \left(\frac{v_i^2}{2\eta} + \frac{\eta}{2} \sum_{t=1}^{T} (g_i^t)^2 + \frac{|v_i| \times 2\delta}{\eta} \right) \\
&\qquad \text{(the diameter in each dimension is now bounded by } 2\delta\text{)} \\
&\leq O\left(\frac{\delta \sum_{i=1}^{d} |v_i|}{\eta} + \eta G^2 T \right) \\
&\leq O\left(\frac{\delta^2 \sqrt{d}}{\eta} + \eta G^2 T \right). \qquad \text{(by Cauchy-Schwarz, } \textstyle\sum_i |v_i| \leq \sqrt{d}\sqrt{\sum_i |v_i|^2} \leq \delta\sqrt{d}\text{)}
\end{aligned}$$

Choosing the optimal $\eta = \frac{d^{\frac{1}{4}}\delta}{G\sqrt{T}}$, we get the regret upper bound of order $O\left(d^{\frac{1}{4}} G \delta \sqrt{T}\right)$.